Recoverable-Support Geometry and Response Visibility: Descent, Holonomy Intertwiners, and the Premetric G1 Handoff

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Finite causal-screen holography treats a bulk distinction as physically accessible relative to a finite screen only when boundary data support its recovery at a declared capacity, tolerance, and verification window. FDS–H1 introduced finite screen ledgers, regional recovery maps, overlap intertwiners, tolerant gluing, and recovery holonomy. H2 asks when that finite recovery structure can support a separately registered response obstruction.

The paper distinguishes raw regional descent, quotient descent and liftability, mixed gluing–transport compatibility, and local connection curvature. Its principal geometric realization is a relative Kato connection on a smooth constant-rank recoverable-support bundle inside a registered ambient Hermitian bundle. This construction is deliberately support-limited: equal support projectors imply equal Kato connections and curvatures even when channel spectra, probabilities, fidelities, or actions differ.

The response side is split into two independent bridge branches. In the operational-covector branch, a directly registered G1 response one-form ω_{th} is related to physical recovery curvature by a pre-registered parallel functional, $\mathrm{d}\omega_{\mathrm{th}} = \ell(F_{\mathrm{phys}})$, and is tested by Stokes-type loop–surface comparisons. A global covector bridge additionally requires the de Rham class $[\ell(F_{\mathrm{phys}})]_{\mathrm{dR}}$ to vanish. In the response-bundle branch, an independently registered response connection is related to recovery geometry by a parallel bundle morphism; existence is equivalent to holonomy intertwining at a base point. These branches have different data, global obstructions, approximation errors, and validation protocols.

Character-based one-dimensional responses see only the Lie-algebra Abelianization of the identity component, whereas general parallel covector readouts obey a weaker holonomy-relative visibility bound. Discrete characters may additionally detect global flat monodromy. A three-qutrit calculation is retained only as a code-seeded candidate-support audit: the correctability-preserving control is flat, while the tested nonzero-support-curvature deformation fails the Knill–Laflamme criterion and is ineligible for physical-quotient analysis. H2 therefore establishes a formal recoverable-support bridge architecture, branch-specific existence criteria, visibility limits, and validation protocols. It does not yet provide a completed nontrivial physical bridge, a full recovery-channel geometry, a curved QEC realization, general relativity, the G1 residual tensor, or the $M_{3/4}$ branch.

Scope and Claim Status. H2 is a restricted mathematical and operational framework connecting H1 finite recovery to the response-obstruction layer of G1. Its formal core consists of recovery descent, registered chamberwise connection rules, regular principal-bundle reduction, two branch-specific bridge criteria, scalar-response visibility bounds, finite-cost non-absorption, and finite-resolution validation.

The Kato branch is explicitly a *recoverable-support geometry.* It detects how a registered constant-rank support subspace is transported relative to a registered ambient Hermitian connection. It does not determine spectral weights, recovery probabilities, channel fidelity, or the full action of a recovery channel inside a fixed support. H2 therefore distinguishes a completed support-geometry branch from an open full-channel branch.

H2 also keeps two response realizations distinct. In the operational-covector branch, ω_{th} is independently registered as a physical one-form and is not treated as a gauge potential. In the response-bundle branch, F_{resp} is the primary gauge-covariant response object and a local potential exists only after a physical response frame is registered. The holonomy-intertwiner theorem applies only to the bundle branch. Failure of either branch-specific criterion demotes that tested H1-to-G1 bridge; it does not falsify H1, G1 as an independent response program, standard holography, general relativity, or the FDS formal core.

Claim-status summary

Keywords: finite distinction systems; finite causal-screen holography; recovery descent; recoverable-support geometry; Kato connection; holonomy; lifting obstruction; physical quotient; Abelianization; screen response; quantum error correction; G1.

TABLE I. Central FDS–H2 claims, status, and demotion conditions.

Claim	**Status**	**Demotion or limitation**
Raw exact descent, quotient descent, and strict liftability are distinct	Restricted formal propositions	A G_0-valued defect guarantees quotient descent only; a strict lift requires triviality of the associated twisted lifting obstruction
A connection must be generated by a pre-registered gauge-covariant rule	Admissibility rule; relative Kato theorem	If curvature changes under equivalent presentations or response-dependent ambient geometry, it is not an identifiable support invariant
Kato transport is a recoverable-support geometry, not a full-channel geometry	Scope theorem and blindness proposition	Equal support projectors imply equal Kato connections and curvatures even when channel spectra, weights, fidelity, or action differ
The physical recovery quotient is a regular principal-bundle quotient	Restricted quotient theorem	Outside the closed-normal, regular branch the quotient must be treated as stratified, orbifold, groupoid, or stack-like
The operational-covector bridge is $\mathrm{d}\omega_{\mathrm{th}} = \ell(F_{\mathrm{phys}})$	Conditional functional bridge and Stokes protocol	A global bridge requires $[\ell(F_{\mathrm{phys}})]_{\mathrm{dR}} = 0$; failure demotes the functional, scale calibration, or global covector realization
A parallel bundle bridge exists exactly when a base-point map intertwines all holonomies	Bundle-branch theorem; standard holonomy principle specialized to registered bundles	Held-out loops support only the sampled subgroup unless generators are proved; this theorem does not apply to the direct covector branch
Parallel Branch-A readouts are holonomy-relative; character-based one-dimensional responses see only the full Lie-algebra Abelianization of the identity component	Holonomy-invariance proposition and character no-go theorem	A general parallel readout annihilates $[\mathfrak{hol}_{X_0}, \mathfrak{g}_{\mathrm{phys}}]$, not necessarily $[\mathfrak{g}_{\mathrm{phys}}, \mathfrak{g}_{\mathrm{phys}}]$. Full Abelianization requires full-group invariance, a character differential, or equality of the holonomy-generated and full commutator subspaces
H2 derives G1, GR, or $M_{3/4}$	Explicitly not claimed	H2 ends at a support-level premetric response-obstruction criterion; full-channel geometry, Ward closure, optical selection, and cosmology remain downstream

INTRODUCTION

The missing H1-to-G1 bridge

H1 treats holography as finite boundary distinction recovery. A screen is not merely a geometric surface but a finite ledger carrying accessible distinctions, recovery redundancy, local reconstruction maps, and finite gluing tolerances [2]. G1, by contrast, formulates finite-screen response on a manifold of screen states. In the operational-covector reading used by G1, its premetric diagnostic is a directly registered response one-form

$$\omega_{\mathrm{th}} = \eta_i(X)\,\mathrm{d}X^i, \tag{1}$$

with operational nonclosure

$$\mathrm{d}\omega_{\mathrm{th}} \neq 0 \tag{2}$$

indicating that the registered response is not generated by a single local scalar potential before later metric, tensor, Ward, or cosmological closure [3].

The analogy between a failed recovery loop and a nonclosed response form is suggestive but insufficient. Regional recovery data live over a cover of a finite screen. The response form lives over a manifold of screen states or control parameters. Even after a recovery bundle is constructed, its connection is not determined by the bundle alone. Even after a physical recovery curvature is defined, a scalar response may be algebraically unable to detect it. H2 addresses these missing steps.

The revised central question

The central question is not merely

$$\text{if } J \text{ preserves connections, does it preserve curvature?} \tag{3}$$

That implication is standard differential geometry. The nontrivial questions are

$$\begin{gathered}\text{Which recovery data fix an admissible connection?}\\ \text{When is the physical quotient regular?}\\ \text{When does a recovery–response map exist?}\\ \text{Which curvature sectors are response-visible?}\end{gathered} \tag{4}$$

Six logical stages and a response fork

The hardened H2 chain is

$$\begin{aligned} &\text{recovery data} \to \text{descent chamber} \\ &\to \text{support connection} \to \text{physical quotient} \\ &\to \begin{cases} \text{covector bridge,} \\ \text{bundle bridge,} \end{cases} \\ &\to \text{G1 candidate.} \end{aligned} \tag{5}$$

Every arrow has a failure condition. No failure may be absorbed by redefining the connection, enlarging the response space without cost, or fitting a response map to the same loops or surfaces used for validation. Table III provides the main visual comparison of the two response branches.

Main contributions

This paper makes twelve contributions.

1. It formulates regional recovery and state-parameter transport as a non-Abelian Čech–de Rham descent hierarchy.

2. It distinguishes raw exact descent, physical quotient descent, and strict liftability through the lifting obstruction of the registered group extension.

3. It defines a registered connection rule and gives the relative Kato recoverable-support connection for constant-rank supports inside registered ambient Hermitian geometry.

4. It proves support-equivalent channel blindness and separates the completed support-geometry branch from an open full-channel branch.

5. It derives the principal transport bundle from the unitary frame bundle and gives a regular principal-bundle quotient theorem for task-defined gauge reduction.

6. It separates an operational-covector bridge from a response-bundle bridge and gives a branch-comparison table.

7. It gives the covector functional criterion $d\omega_{\text{th}} = \ell(F_{\text{phys}})$, its Stokes validation protocol, and its global de Rham obstruction.

8. It specializes the standard holonomy principle to the bundle branch, yielding existence and identifiability criteria for a parallel recovery–response morphism.

9. It treats bundle curvature naturality as a standard consequence and derives the exact Hom-bundle defect identity for approximate bundle bridges.

10. It proves holonomy-relative blindness for general parallel flat-coefficient readouts, character-based scalar Abelianization, and observable-kernel blindness, including the discrete-character exception.

11. It introduces finite-cost non-absorption, branch-specific finite-resolution tests, and dimensional and normalization registration.

12. It provides a code-seeded candidate-support audit and registers curved correctable code manifolds as an open existence-or-no-go problem.

INPUTS AND NON-IDENTIFICATION RULES

H1 finite recovery input

For a screen state X, let

$$\Sigma_X = \bigcup_{a\in I} A_a(X) \tag{6}$$

be a finite admissible cover. For each region $A_a(X)$, H1 supplies a task-relative recoverable sector $\mathcal{Q}_a(X)$, a local recovery map

$$R_a(X) : \mathcal{B}_{A_a(X)} \longrightarrow \mathcal{Q}_a(X), \tag{7}$$

and overlap comparisons between locally recovered representatives [2]. H2 imports this architecture; it does not rederive it.

Prior-art positioning. Parrikar et al. construct relational reconstruction flow along smooth families of code subspaces that remain correctable against a fixed erasure [21]. H2 addresses a different layer: regional descent, task-defined physical quotienting, and branch-specific maps from recoverable-support curvature to independently registered responses. Modern operator-algebra, recoverability, and approximate-QEC results emphasize that support information alone does not establish task-relative correctability or recovery performance [22–24, 26]. Accordingly, H2 requires an independent correctability audit before a code-seeded support family is admitted to the physical recovery quotient.

G1 response input

G1 supplies an independently registered response structure on a screen-state space $\mathcal{M}_{\text{scr}}$. H2 does not define the G1 response by inserting a recovery connection into a formula. The response bundle, response connection, or thermodynamic response form must be fixed independently and then tested against recovery transport.

Two base structures

The regional labels and state-space labels must not be identified:

$$\begin{aligned} a, b, c &: \text{ screen-region labels,} \\ i, j, k &: \text{ directions in } \mathcal{M}_{\rm scr}. \end{aligned} \tag{8}$$

Regional intertwiners compare two reconstructions at fixed X. Parameter transport compares recovery sectors at nearby X.

Q0 non-identification

Q0 boundary-access holonomy follows directed system–apparatus–environment–record–recovery loops [4]. H2 recovery holonomy follows a family of recoverable sectors over screen regions and screen-state parameters. Specific models may relate them, but H2 does not identify them by definition.

Imported regularity assumptions

Smooth finite-dimensional chambers, locally constant rank, and regular Lie-group actions are imported mathematical assumptions. They are not consequences of the FDS distinction primitive. Rank-changing, non-Hausdorff, or nonregular sectors are retained as demotion branches rather than forced into the smooth theory.

NON-ABELIAN RECOVERY DESCENT HIERARCHY

Parameterized recovery atlas

Definition 1 (Parameterized recovery atlas)**.** *A parameterized recovery atlas on $U \subset \mathcal{M}_{\rm scr}$ consists of local recoverable sectors $\mathcal{Q}_a(X)$, local recovery maps $R_a(X)$, and invertible overlap intertwiners*

$$g_{ab}(X) : \mathcal{Q}_a(X) \longrightarrow \mathcal{Q}_b(X) \tag{9}$$

for registered common distinctions on $A_a \cap A_b$.

The intertwiners compare two local recovery conventions on a common task sector. They are not inferred from the target response.

Raw descent, quotient descent, and strict liftability

On a triple overlap define the group-valued defect

$$h_{abc}(X) = g_{ca}(X) g_{bc}(X) g_{ab}(X). \tag{10}$$

Let

$$1 \longrightarrow G_0 \longrightarrow G_{\rm tr} \xrightarrow{q} G_{\rm phys} \longrightarrow 1 \tag{11}$$

be the registered group extension on a regular branch.

Definition 2 (Three descent branches)**.** *The regional atlas has:*

1. raw exact descent *if $h_{abc} = e$;*
2. quotient descent *if $q(h_{abc}) = e$, equivalently $h_{abc} \in G_0$;*
3. *a* strictly liftable quotient atlas *if quotient descent holds and there exists an admissible G_0-valued one-cochain n_{ab} such that the modified lifts*

$$g'_{ab} = n_{ab} g_{ab} \tag{12}$$

satisfy $g'_{ca} g'_{bc} g'_{ab} = e$.

Definition 3 (Recovery lifting obstruction)**.** *For quotient transitions $\bar{g}_{ab} = q(g_{ab})$, chosen lifts g_{ab} define a G_0-valued triple-overlap defect h_{abc}. If the lifts are changed by a G_0-valued one-cochain,*

$$g'_{ab} = n_{ab} g_{ab}, \qquad n_{ab} : U_{ab} \to G_0, \tag{13}$$

then, with ${}^g n := g n g^{-1}$,

$$h'_{abc} = n_{ca} \left({}^{g_{ca}} n_{bc}\right) \left({}^{g_{ca} g_{bc}} n_{ab}\right) h_{abc}. \tag{14}$$

Strict liftability requires a solution of the twisted non-Abelian cochain equation

$$\boxed{n_{ca} \left({}^{g_{ca}} n_{bc}\right) \left({}^{g_{ca} g_{bc}} n_{ab}\right) h_{abc} = e.} \tag{15}$$

The obstruction to solving Eq. (15) is the lifting obstruction associated with the extension in Eq. (11). This is generally described by twisted non-Abelian descent, crossed modules, or a lifting-gerbe construction rather than an ordinary Abelian H^2 class [27–31].

Proposition 1 (Raw descent, quotient descent, and liftability)**.** *Fix a chamber in which common recovery rank and orbit type are locally constant.*

1. *If $h_{abc} = e$, the raw transition maps define an ordinary $G_{\rm tr}$ recovery bundle.*
2. *If $q(h_{abc}) = e$, the quotient transitions $\bar{g}_{ab} = q(g_{ab})$ define a $G_{\rm phys}$ bundle, provided the quotient is regular.*
3. *The condition $h_{abc} \in G_0$ does not imply strict liftability. A strict raw lift exists only when the associated lifting obstruction is trivial.*

Proof. The first statement is the ordinary cocycle gluing theorem. For the second,

$$q(g_{ca})q(g_{bc})q(g_{ab}) = q(h_{abc}) = e, \tag{16}$$

so the quotient transitions satisfy a cocycle. For the third, changing local frames by a zero-cochain conjugates the defect but does not generally eliminate it. A different choice of lifts is encoded by a G_0-valued one-cochain n_{ab}; the modified lifts form a strict cocycle only when Eq. (15) admits a solution. This is stronger than quotient descent. □

A non-liftable quotient atlas may require a lifting-gerbe, crossed-module, groupoid, or stack-like description. H2 records the distinction but does not develop the full higher-descent formalism.

Parameter transport and mixed defect

Within each regional chart, a parameter transport is

$$U^{(a)}_{X+\mathrm{d}X,X} = I + \Gamma^{(a)}_i(X)\mathrm{d}X^i + O(\|\mathrm{d}X\|^2). \tag{17}$$

The mixed defect is the Hom-bundle-valued one-form

$$K_{ab} = \mathrm{d}g_{ab} + \Gamma^{(b)}g_{ab} - g_{ab}\Gamma^{(a)}. \tag{18}$$

It measures the failure of regional comparison and parameter transport to commute.

Proposition 2 (Connection descent). *Assume raw exact descent, or work on the regular quotient bundle after quotient descent. The local connection forms define one global connection if and only if $K_{ab} = 0$ on every overlap.*

Curvature compatibility

Let

$$F_a = \mathrm{d}\Gamma^{(a)} + \Gamma^{(a)} \wedge \Gamma^{(a)}. \tag{19}$$

Regard K_{ab} as a one-form valued in $\mathrm{Hom}(E_a, E_b)$. Its Hom-bundle covariant derivative is

$$\nabla^{\mathrm{Hom}}K_{ab} = \mathrm{d}K_{ab} + \Gamma^{(b)} \wedge K_{ab} + K_{ab} \wedge \Gamma^{(a)}. \tag{20}$$

Direct calculation gives

$$\nabla^{\mathrm{Hom}}K_{ab} = F_b g_{ab} - g_{ab}F_a. \tag{21}$$

Hence $K_{ab} = 0$ implies $F_b = g_{ab}F_a g_{ab}^{-1}$. The triplet

$$\boxed{(h_{abc}, K_{ab}, F_a)} \tag{22}$$

is the non-Abelian Čech–de Rham recovery descent hierarchy. It is not asserted to be an ordinary linear bicomplex in the fully non-Abelian branch.

RECOVERY CHAMBERS AND REGISTERED CONNECTION RULES

Constant-rank chambers

Definition 4 (Recovery chamber). *A recovery chamber $\mathcal{M}_\alpha \subset \mathcal{M}_{\mathrm{scr}}$ is a connected region on which the recoverable rank, support type, orbit type, gauge subgroup, and response representation are constant, the regional intertwiners remain locally invertible, and no capacity-crossing or decoder-branch transition occurs.*

A recovery phase boundary occurs when, for example,

$$r(X^-) \neq r(X^+). \tag{23}$$

The smooth theory is chamberwise. Approximate-code thresholds and decoder transitions motivate treating tolerance and decoder-branch crossings as chamber boundaries [24].

Why the bundle does not determine the connection

A vector bundle admits many connections. H2 therefore requires a connection rule to be registered before response fitting.

Definition 5 (Registered connection rule). *A registered connection rule $\mathfrak{C}$ maps admissible recovery data, together with all required ambient comparison structure, to a connection:*

$$\mathfrak{C} : \{\textit{recovery and ambient data}\} \longrightarrow \Gamma_{\mathrm{tr}}. \tag{24}$$

It must be local, smooth within the chamber, gauge covariant, stable under equivalent presentations, and independent of held-out response loops.

Relative Kato recoverable-support connection

Let $(\mathcal{H}_{\mathrm{amb}}, \langle\cdot,\cdot\rangle, \nabla^{\mathrm{amb}})$ be a registered ambient Hermitian bundle with unitary connection and curvature $F_{\mathrm{amb}} = (\nabla^{\mathrm{amb}})^2$. Let

$$P(X)^2 = P(X) = P(X)^\dagger, \qquad \mathrm{rank}\, P(X) = r, \tag{25}$$

be a smooth orthogonal projector, and let $E = \mathrm{Im}\, P \subset \mathcal{H}_{\mathrm{amb}}$.

Theorem 1 (Relative Kato recoverable-support connection). *The support bundle E carries the projected connection*

$$\nabla^K = P\nabla^{\mathrm{amb}}. \tag{26}$$

Its curvature is

$$\boxed{F_K = PF_{\mathrm{amb}}P + P(\nabla^{\mathrm{amb}}P \wedge \nabla^{\mathrm{amb}}P)P.} \tag{27}$$

This is the standard Kato projected-connection construction specialized to a registered recoverable support. It is canonical relative to the registered ambient Hermitian bundle, metric, and connection. The projector-only expression is the flat-ambient special case [8, 9, 11].

Proposition 3 (Support-equivalent channel blindness)**.** *Let $\mathcal{R}_1(X)$ and $\mathcal{R}_2(X)$ be smooth constant-rank recovery-channel families represented in the same registered ambient Hermitian bundle with the same ambient connection. If*

$$P_{\mathcal{R}_1}(X) = P_{\mathcal{R}_2}(X) \quad \forall X \in \mathcal{M}_\alpha, \tag{28}$$

then

$$\nabla^K_{\mathcal{R}_1} = \nabla^K_{\mathcal{R}_2}, \qquad F^K_{\mathcal{R}_1} = F^K_{\mathcal{R}_2}. \tag{29}$$

This remains true even when the two channels have different Choi eigenvalues, recovery probabilities, fidelities, or actions within the common support.

Proof. Equations (26) and (27) depend only on the common projector and the registered ambient connection. Internal positive weights and channel action on the same support do not enter the Kato construction. □

The word "canonical" is therefore relative and support-limited. An ambient frame or connection may not be selected after observing the target response, and equality of Kato geometry does not imply equality of recovery performance.

Support-geometry and full-channel branches

H2 separates two model classes. The completed branch is

$$\text{support branch:} \qquad P_{\mathcal{R}},\ \nabla^K,\ F_K. \tag{30}$$

The open branch retains the complete channel data,

$$\text{full-channel branch:} \qquad C_{\mathcal{R}},\ \operatorname{spec} C_{\mathcal{R}}, \tag{31}$$

together with weights, fidelity, and channel action. The present paper completes only the first branch. Candidate full-channel geometries include Choi-positive-operator information geometry, Uhlmann/Bures transport, and Petz-family transport, but no universal full-channel connection is claimed here. Recent work has represented quantum-channel spaces as quotients of complex Stiefel manifolds and studied the resulting geometric and optimization structures [25]; Jencová characterizes channel sufficiency via hypothesis testing and L_1-distance preservation [23]. The Kato construction used in H2 is deliberately narrower: it retains only constant-rank support transport relative to registered ambient geometry.

From support bundle to principal transport bundle

Let $\mathrm{Fr}_U(E) \to \mathcal{M}_\alpha$ be the unitary frame bundle of E. It is a principal $U(r)$-bundle, and the relative Kato connection induces a principal $U(r)$-connection Γ_{tr}.

Proposition 4 (Frame-bundle induction and reduction)**.** *In the Kato branch one may take $G_{\mathrm{tr}} = U(r)$. A closed subgroup $G_{\mathrm{tr}} \subseteq U(r)$ defines an admissible reduced transport bundle only when the Kato connection preserves the reduction, equivalently when its horizontal distribution is tangent to the reduced frame bundle or its connection form restricts to $\mathfrak{g}_{\mathrm{tr}}$.*

Minimum-change connection

A second model class defines transport by

$$U^\star_{X+\mathrm{d}X,X} = \arg\min_U \left[D_{\mathrm{task}}(R_{X+\mathrm{d}X} \circ U, R_X) + \lambda C(U)\right]. \tag{32}$$

This rule is admissible only if existence, local uniqueness, Hessian nondegeneracy, gauge covariance, and independence from validation data are proved.

Criterion 1 (No response-selected connection)**.** *A connection rule is inadmissible if its ambient geometry, support selection, gauge convention, transport cost, or hyperparameters are chosen using the same response loops later used to claim the bridge.*

REGULAR PHYSICAL RECOVERY QUOTIENT

Task-defined gauge subgroup

Let $P_{\mathrm{tr}} \to \mathcal{M}_\alpha$ be the registered principal G_{tr}-bundle of admissible recovery frames; in the Kato branch it is the full or an admissible reduced unitary frame bundle.

Definition 6 (Recovery gauge subgroup)**.** $G_0 \subset G_{\mathrm{tr}}$ *consists of frame transformations that leave the registered distinction algebra, task predictions, recovery loss, and declared screen observables invariant.*

Theorem 2 (Regular physical quotient)**.** *Assume $G_0 \triangleleft G_{\mathrm{tr}}$ is a closed normal Lie subgroup. On a regular chamber satisfying the smooth-quotient hypotheses,*

$$P^{\mathrm{phys}}_{\mathrm{rec}} = P_{\mathrm{tr}}/G_0 \tag{33}$$

is a principal $G_{\mathrm{phys}} = G_{\mathrm{tr}}/G_0$-bundle. If Γ_{tr} is a genuine G_{tr}-connection, its projection descends to Γ_{phys}, with curvature

$$F_{\mathrm{phys}} = \mathrm{d}\Gamma_{\mathrm{phys}} + \Gamma_{\mathrm{phys}} \wedge \Gamma_{\mathrm{phys}}. \tag{34}$$

TABLE II. Information retained and lost along the H2 visibility chain.

Layer	Retained structure	Quotiented or lost structure	H2 status
Full recovery channel	support, spectrum, weights, action, and fidelity	none by definition	open geometric branch
Kato support geometry	support transport relative to ambient connection	spectrum, weights, fidelity, and internal channel action	completed restricted branch
Physical support quotient	task-visible support curvature	registered gauge directions	regular-quotient theorem
Response-visible curvature	image under the registered response representation	representation-kernel sector	bridge-dependent output

This is the standard regular principal-quotient theorem specialized to the task-defined physical gauge. The theorem concerns quotient descent. It does not imply that a raw G_0-valued regional cocycle defect is removable before quotienting.

Irregular branches

If the subgroup is not closed or normal, orbit type changes, or the quotient is singular, H2 records an irregular branch requiring an orbibundle, stratified quotient, groupoid, or quotient-stack description.

INDEPENDENT RESPONSE GEOMETRY AND TWO BRIDGE BRANCHES

H2 does not redefine the primitive response object of G1. It develops two distinct bridge realizations with different mathematical inputs, global obstructions, and validation protocols.

Branch A: Operational Covector Bridge with Flat Coefficients

Let V_{cov} be a fixed finite-dimensional real vector space and let

$$\underline{V}_{\mathrm{cov}} = \mathcal{M}_\alpha \times V_{\mathrm{cov}} \tag{35}$$

denote the associated trivial coefficient bundle with its standard flat connection

$$\nabla^{\underline{V}} = \mathrm{d}. \tag{36}$$

The scalar G1 response corresponds to $V_{\mathrm{cov}} = \mathbb{R}$; a finite multicomponent response corresponds to $V_{\mathrm{cov}} = \mathbb{R}^m$.

The independently registered operational response is a V_{cov}-valued one-form

$$\omega_{\mathrm{th}} \in \Omega^1(\mathcal{M}_\alpha; \underline{V}_{\mathrm{cov}}) \simeq \Omega^1(\mathcal{M}_\alpha) \otimes V_{\mathrm{cov}}. \tag{37}$$

It is a physical covector, not a gauge potential. Let

$$\ell \in \Gamma\big[\mathrm{Hom}\big(\mathrm{ad}(P^{\mathrm{phys}}_{\mathrm{rec}}), \underline{V}_{\mathrm{cov}}\big)\big] \tag{38}$$

be a pre-registered fiberwise-linear readout. The physical adjoint connection and the flat target connection induce a Hom-bundle connection

$$\nabla^{\mathrm{Hom}} = \nabla^{\underline{V}} \otimes 1 + 1 \otimes \nabla^{(\mathrm{ad}\, P^{\mathrm{phys}}_{\mathrm{rec}})^*}. \tag{39}$$

Equivalently, for a section s of $\mathrm{ad}(P^{\mathrm{phys}}_{\mathrm{rec}})$,

$$(\nabla^{\mathrm{Hom}}\ell)(s) = \nabla^{\underline{V}}(\ell(s)) - \ell(D_{\mathrm{phys}} s). \tag{40}$$

The clean compatibility condition is

$$\boxed{\nabla^{\mathrm{Hom}}\ell = 0.} \tag{41}$$

Because $D_{\mathrm{phys}} F_{\mathrm{phys}} = 0$ and $\nabla^{\mathrm{Hom}}\ell = 0$,

$$\mathrm{d}\,\ell(F_{\mathrm{phys}}) = (\nabla^{\mathrm{Hom}}\ell)(F_{\mathrm{phys}}) + \ell(D_{\mathrm{phys}} F_{\mathrm{phys}}) = 0. \tag{42}$$

The operational-covector bridge is therefore

$$\boxed{\mathrm{d}\omega_{\mathrm{th}} = \ell(F_{\mathrm{phys}}).} \tag{43}$$

The dimensions, normalization, coordinate convention, and calibration constants of ℓ are fixed before validation.

This branch uses the ordinary exterior derivative, the ordinary vector-valued Stokes theorem, and ordinary de Rham cohomology with coefficients in the fixed vector space V_{cov}. A nontrivial or non-flat response coefficient bundle is not included in this covector branch; it belongs to a future covariant-covector extension or to the response-bundle branch.

Remark 1 (Holonomy-relative invariance)**.** *The condition $\nabla^{\mathrm{Hom}}\ell = 0$ implies that ℓ_{X_0} is invariant under conjugation by the registered holonomy group $\mathrm{Hol}_{X_0}(\Gamma_{\mathrm{phys}})$, not necessarily under the full structure group G_{phys}. Hence ℓ_{X_0} annihilates $[\mathfrak{hol}_{X_0}, \mathfrak{g}_{\mathrm{phys}}]$. Full G_{phys}-Abelianization follows only if (i) ℓ extends to an $\mathrm{Ad}(G_{\mathrm{phys}})$-invariant functional, (ii) in the one-dimensional case $\ell = \rho_*$ is the differential of a group character, or (iii) $[\mathfrak{hol}_{X_0}, \mathfrak{g}_{\mathrm{phys}}] = [\mathfrak{g}_{\mathrm{phys}}, \mathfrak{g}_{\mathrm{phys}}]$. A sufficient stronger condition for (iii) is $\mathfrak{hol}_{X_0} = \mathfrak{g}_{\mathrm{phys}}$. The Bianchi closure argument $\mathrm{d}\,\ell(F_{\mathrm{phys}}) = 0$ remains valid in all cases and does not require full-group invariance.*

Proposition 5 (Global covector-bridge obstruction)**.** *If a globally defined response one-form $\omega_{\rm th}$ satisfies Eq. (43), then*

$$[\ell(F_{\rm phys})]_{\rm dR} = 0 \quad in \quad H^2_{\rm dR}(\mathcal{M}_\alpha; V_{\rm cov}). \qquad (44)$$

For fixed finite-dimensional coefficients, $H^2_{\rm dR}(\mathcal{M}_\alpha; V_{\rm cov}) \simeq H^2_{\rm dR}(\mathcal{M}_\alpha) \otimes V_{\rm cov}$. Conversely, if the relevant vector-valued de Rham class vanishes, a global one-form solution exists up to addition of a closed $V_{\rm cov}$-valued one-form. On a contractible chart a local solution exists whenever $\ell(F_{\rm phys})$ is closed.

Proof. The exterior derivative of a global $V_{\rm cov}$-valued one-form is exact, proving necessity componentwise. Vanishing of the vector-valued de Rham class is precisely the existence condition for a global primitive. Local existence follows from the componentwise Poincaré lemma. □

A nontrivial class in Eq. (44) does not make the recovery curvature unphysical. It means that one global operational response one-form cannot realize the proposed bridge, even though local potentials may exist.

Non-uniqueness. When global primitives exist, their set is an affine space modeled on the closed $V_{\rm cov}$-valued one-forms $Z^1_{\rm dR}(\mathcal{M}_\alpha; V_{\rm cov})$. If exact changes $\omega_{\rm th} \mapsto \omega_{\rm th} + {\rm d}\phi$ are admitted as physically equivalent recalibrations, the residual global ambiguity is classified by $H^1_{\rm dR}(\mathcal{M}_\alpha; V_{\rm cov})$. In H2 the response one-form is independently registered rather than reconstructed solely from curvature, so this ambiguity is not a defect of the present bridge.

Protocol 1 (Covector bridge validation)**.** *Register $\omega_{\rm th}$, ℓ, units, normalization, orientation, a calibration family, and held-out surface and cycle families before examining validation data. Use calibration data only to estimate pre-declared constants in ℓ, then freeze the functional and all normalization choices.*

A1. Boundary-surface Stokes residual. *On held-out oriented surfaces Σ_m with boundary, compare*

$$\oint_{\partial\Sigma_m} \omega_{\rm th} \quad with \quad \int_{\Sigma_m} \ell(F_{\rm phys}), \qquad (45)$$

and report

$$\epsilon_{\Sigma_m} = \left\| \oint_{\partial\Sigma_m} \omega_{\rm th} - \int_{\Sigma_m} \ell(F_{\rm phys}) \right\|_{V_{\rm cov}}. \qquad (46)$$

This tests the local differential bridge, finite-resolution convergence, orientation conventions, and normalization.

A2. Closed-cycle period test. *For a declared family of closed two-cycles C_r and, when available, a generating set representing $H_2(\mathcal{M}_\alpha; \mathbb{R})$, compute*

$$\Pi_{C_r} = \int_{C_r} \ell(F_{\rm phys}) \in V_{\rm cov}. \qquad (47)$$

If any registered period is nonzero, a single globally defined operational response one-form cannot realize Eq. (43). Finite tests establish this conclusion only on the declared cycle family; if the tested cycles are proved to generate the relevant second homology, the result upgrades to a global de Rham obstruction.

Report the boundary-surface residuals and closed-cycle periods separately. A local Stokes fit does not by itself establish global exactness.

The differential covector residual is

$$\Delta_{\rm cov} = {\rm d}\omega_{\rm th} - \ell(F_{\rm phys}). \qquad (48)$$

This branch requires no response bundle, response parallel transport, or morphism J.

Branch B: response-bundle bridge

In the second branch, let $P_{\rm resp} \to \mathcal{M}_\alpha$ be an independently specified principal response bundle with structure group $G_{\rm resp}$ and registered connection

$$A_{\rm resp}. \qquad (49)$$

Its curvature is

$$F_{\rm resp} = {\rm d}A_{\rm resp} + A_{\rm resp} \wedge A_{\rm resp}. \qquad (50)$$

Within this branch, $F_{\rm resp}$ is the primary gauge-covariant response-side object. The response connection is calibrated from response-side variables and is not defined from $\Gamma_{\rm phys}$.

Physical response-frame criterion in the bundle branch

Criterion 2 (Physical response frame)**.** *A local section s of the response principal bundle is physically registered only when:*

1. *it is determined by a response-side calibration protocol, fixed reference response state, boundary-record standard, global observable frame, or a pre-registered unique optimization rule;*
2. *it is chosen without access to held-out recovery loops;*
3. *it is locally unique up to a stabilizer that leaves every registered response observable invariant;*
4. *it obeys the registered overlap-patching law; and*
5. *changing among admissible representatives does not alter the curvature-level conclusion.*

For such a section,

$$\omega_{\mathrm{loc}} = s^* A_{\mathrm{resp}} \tag{51}$$

is a locally registered response potential. In an Abelian response sector,

$$\mathrm{d}\omega_{\mathrm{loc}} = s^* F_{\mathrm{resp}} \tag{52}$$

locally, with the usual overlap and global-cohomology qualifications. The symbol ω_{th} remains reserved for the independently registered operational covector unless an additional equivalence is explicitly established.

Branch comparison

Scale calibration and registration discipline

A physical comparison with G1 may require

$$\Omega_{\mathrm{G1}} = \lambda_{\mathrm{cal}}\, \mathcal{N}_X[\Omega_{\mathrm{rec}}], \tag{53}$$

where Ω_{rec} denotes either $\ell(F_{\mathrm{phys}})$ or a registered component of F_{resp}. The calibration λ_{cal}, coordinate-dependent normalization $\mathcal{N}_X$, units, and reparameterization rule are registered before validation. The operational covector, response bundle, A_{resp}, physical section, ℓ, and scale calibration may be fixed using theory, calibration data, or a disjoint training set. Validation loops or surfaces may not be reused to choose them.

BUNDLE-BRANCH HOLONOMY CRITERION

This section applies only to the response-bundle branch.

Parallel morphisms

Let $E_{\mathrm{rec}}^{\mathrm{phys}}$ and E_{resp} be associated vector bundles with connections ∇^{phys} and ∇^{resp}. A candidate bundle bridge is a fiberwise-linear, constant-rank bundle map

$$J : E_{\mathrm{rec}}^{\mathrm{phys}} \longrightarrow E_{\mathrm{resp}} \tag{54}$$

covering the identity on $\mathcal{M}_\alpha$. It is parallel if

$$\nabla^{\mathrm{resp}} \circ J = J \circ \nabla^{\mathrm{phys}}. \tag{55}$$

Bridge-existence theorem

Theorem 3 (Bundle-branch holonomy-intertwiner existence)**.** *Let $\mathcal{M}_\alpha$ be connected, choose a base point X_0, and let $U_{\mathrm{rec}}(\gamma)$ and $U_{\mathrm{resp}}(\gamma)$ denote parallel transport along a path γ. For a fiberwise-linear map*

$$J_0 : E_{\mathrm{rec},X_0}^{\mathrm{phys}} \longrightarrow E_{\mathrm{resp},X_0}, \tag{56}$$

the following are equivalent:

1. *there exists a parallel bundle morphism J with $J_{X_0} = J_0$;*
2. *for every closed loop γ based at X_0,*

$$J_0 U_{\mathrm{rec}}(\gamma) = U_{\mathrm{resp}}(\gamma) J_0. \tag{57}$$

When these conditions hold,

$$J_X = U_{\mathrm{resp}}(\gamma_{X_0X}) J_0 U_{\mathrm{rec}}(\gamma_{X_0X})^{-1} \tag{58}$$

is path independent. For fixed J_0, the extension is unique.

Proof. If J is parallel, it commutes with parallel transport. Conversely, define J_X by Eq. (58). Two paths differ by a based loop, and Eq. (57) makes the definitions agree. Smoothness and parallelity follow from smooth parallel transport. □

This is the standard parallel-homomorphism/holonomy principle specialized to independently registered finite-recovery and response bundles. H2's contribution lies in constructing and auditing the two sides and in assigning bridge failure to the correct demotion branch.

Identifiability and held-out validation

The allowed base-point maps form

$$\mathrm{Hom}_{\mathrm{Hol}}\left(E_{\mathrm{rec},X_0}^{\mathrm{phys}}, E_{\mathrm{resp},X_0}\right). \tag{59}$$

If this space has dimension greater than one, H2 reports nonidentifiability rather than choosing a preferred J post hoc.

Protocol 2 (Bundle/holonomy bridge validation)**.**

1. *Fix the recovery connection, physical quotient, response bundle, response connection, and loop family.*
2. *Use calibration loops to estimate J_0 and freeze all hyperparameters.*
3. *Test Eq. (57) on disjoint held-out loops.*
4. *Report the dimension and conditioning of the estimated intertwiner space.*
5. *Reject the tested bundle bridge when held-out error exceeds the registered tolerance.*

Finite held-out loops provide evidence only on the sampled holonomy subgroup unless they are proved to generate the registered holonomy group. The Ambrose–Singer principle may be used for the identity component by testing a generating set of parallel-transported curvature values; discrete components require separate loop generators.

TABLE III. The two H2 response bridges are logically distinct.

Item	Operational-covector branch	Response-bundle branch
Response input	$\omega_{\rm th}$	$A_{\rm resp}, F_{\rm resp}$
Bridge	$d\omega_{\rm th} = \ell(F_{\rm phys})$	$\nabla^{\rm resp} J = J\nabla^{\rm phys}$
Validation	loop–surface Stokes test	recovery/response holonomy intertwining
Approximate residual	$\Delta_{\rm cov}$ or ϵ_Σ	$B = \nabla^{\rm Hom} J$
Global obstruction	de Rham exactness	holonomy-representation mismatch
Failure meaning	functional, scale, or global covector failure	morphism or response-representation failure
Response bundle required	no	yes
Parallel morphism J required	no	yes

BUNDLE-BRANCH CURVATURE NATURALITY AND APPROXIMATE DEFECT

Lemma 1 (Curvature naturality)**.** *If J is parallel, then*

$$F_{\rm resp} J = J F_{\rm phys}. \tag{60}$$

Proof. Apply the square of the induced Hom-bundle connection to J. ☐

Define the bundle-bridge defect

$$B = \nabla^{\rm Hom} J = \nabla^{\rm resp} J - J\nabla^{\rm phys}. \tag{61}$$

The exact identity is

$$\boxed{F_{\rm resp} J - J F_{\rm phys} = \nabla^{\rm Hom} B.} \tag{62}$$

This section has no direct covector-branch analogue; the covector residual is Eq. (48).

HOLONOMY-RELATIVE VISIBILITY AND CHARACTER-BASED ABELIANIZATION

Holonomy-relative flat-coefficient readouts

Proposition 6 (Holonomy-relative blindness for parallel flat-coefficient readouts)**.** *Let*

$$\ell \in \Gamma\big[\mathrm{Hom}\big(\mathrm{ad}(P^{\rm phys}_{\rm rec}), \underline{V}_{\rm cov}\big)\big] \tag{63}$$

satisfy $\nabla^{\rm Hom}\ell = 0$. At a base point X_0,

$$\ell_{X_0} \circ \mathrm{Ad}_h = \ell_{X_0} \qquad \forall h \in \mathrm{Hol}_{X_0}(\Gamma_{\rm phys}). \tag{64}$$

Let $\mathfrak{hol}_{X_0} = \mathrm{Lie}(\mathrm{Hol}^0_{X_0}(\Gamma_{\rm phys}))$ be the restricted holonomy algebra. Then

$$\boxed{\ell_{X_0}\big([\mathfrak{hol}_{X_0}, \mathfrak{g}_{\rm phys}]\big) = 0.} \tag{65}$$

Consequently ℓ_{X_0} factors through the vector-space quotient

$$\frac{\mathfrak{g}_{\rm phys}}{[\mathfrak{hol}_{X_0}, \mathfrak{g}_{\rm phys}]}, \qquad [\mathfrak{hol}, \mathfrak{g}] := \mathrm{span}\{[A,B] : A \in \mathfrak{hol},\ B \in \mathfrak{g}\}. \tag{66}$$

No Lie-algebra structure on this quotient is assumed unless the blindness subspace is additionally shown to be an ideal. It need not factor through the full Abelianization $\mathfrak{g}^{\rm ab}_{\rm phys} = \mathfrak{g}_{\rm phys}/[\mathfrak{g}_{\rm phys}, \mathfrak{g}_{\rm phys}]$ without an additional full-group invariance condition.

Proof. Parallel transport in the Hom bundle around a based loop must return the value of the global parallel section to itself. Because the target coefficient bundle is trivial and flat, its holonomy is the identity. Hence $\ell_{X_0} \circ \mathrm{Ad}_h = \ell_{X_0}$ for every $h \in \mathrm{Hol}_{X_0}(\Gamma_{\rm phys})$. Let $A \in \mathfrak{hol}_{X_0}$ and $B \in \mathfrak{g}_{\rm phys}$. Differentiating $\ell_{X_0}(\mathrm{Ad}_{\exp(tA)} B) = \ell_{X_0}(B)$ at $t = 0$ gives $\ell_{X_0}([A,B]) = 0$. Linearity proves the stated annihilation and quotient factorization. ☐

Under a change of base point along a registered path, the holonomy algebra is transported by conjugation and the parallel readout is transported by the induced dual action. Hence the blindness subspace and its vector-space quotient are carried isomorphically to the new base point; the statement is base-point covariant rather than tied to a preferred X_0.

One-dimensional representations and character-based responses

The theorem itself concerns character-based one-dimensional responses. Its Lie-algebra Abelianization conclusion also applies to a readout invariant under the full $\mathrm{Ad}(G_{\rm phys})$ action, although such a functional need not integrate to a global group character. It does not apply automatically to a general parallel Branch-A readout, whose immediate invariance group is only the registered holonomy group; see Remark 1 and Proposition 6.

Theorem 4 (Scalar-response Abelianization)**.** *Let $\rho : G_{\rm phys} \to \mathbb{C}^\times$ or $U(1)$ be a smooth one-dimensional representation with differential*

$$\rho_* : \mathfrak{g}_{\rm phys} \longrightarrow \mathbb{C} \text{ or } i\mathbb{R}. \tag{67}$$

Then

$$\rho_*([A,B]) = 0 \tag{68}$$

for all $A, B \in \mathfrak{g}_{\text{phys}}$. Hence ρ_* factors through the Abelianization

$$\mathfrak{g}_{\text{phys}}^{\text{ab}} = \frac{\mathfrak{g}_{\text{phys}}}{[\mathfrak{g}_{\text{phys}}, \mathfrak{g}_{\text{phys}}]}. \tag{69}$$

If

$$F_{\text{phys}} \in \Omega^2(\mathcal{M}_\alpha, [\mathfrak{g}_{\text{phys}}, \mathfrak{g}_{\text{phys}}]), \tag{70}$$

then

$$\rho_*(F_{\text{phys}}) = 0. \tag{71}$$

Proof. The target Lie algebra is Abelian, so

$$\rho_*([A, B]) = [\rho_*(A), \rho_*(B)] = 0. \tag{72}$$

The quotient statement follows from the universal property of the Abelianization. □

Corollary 1 (Semisimple scalar blindness)**.** *If $\mathfrak{g}_{\text{phys}}$ is perfect, in particular for a semisimple algebra with $[\mathfrak{g}_{\text{phys}}, \mathfrak{g}_{\text{phys}}] = \mathfrak{g}_{\text{phys}}$, every smooth one-dimensional Lie-algebra character vanishes. Pure semisimple recovery curvature cannot be read by a scalar associated connection.*

Remark 2 (Discrete character exception)**.** *The theorem constrains local curvature in the identity component. If G_{phys} is disconnected, a one-dimensional group character may have zero differential while acting nontrivially on discrete components. Such a character can detect global flat monodromy even though it detects no infinitesimal curvature. This is a global-sector effect, not an exception to local Abelianization.*

Consequences for H2 and G1

A general parallel Branch-A readout factors through the holonomy-relative vector-space quotient

$$\frac{\mathfrak{g}_{\text{phys}}}{[\mathfrak{hol}_{X_0}, \mathfrak{g}_{\text{phys}}]}. \tag{73}$$

It may therefore detect directions lying in the semisimple part of the full structure algebra, provided those directions survive this quotient.

A character-based one-dimensional response, or a readout that extends to a full $\mathrm{Ad}(G_{\text{phys}})$-invariant functional, is more restrictive and factors through the full Lie-algebra Abelianization. Such a response can detect:

- a nontrivial central or Abelian curvature component;
- an independently justified nonlinear invariant not represented as a one-dimensional connection;
- a scalar projection of a multicomponent response after that response has first been physically constructed.

It cannot directly detect a pure commutator curvature by assigning an arbitrary nonzero value to that commutator.

Representation-kernel blindness

Theorem 5 (Observable-kernel blindness)**.** *Let $\rho_{\mathcal{O}*} : \mathfrak{g}_{\text{phys}} \to \mathfrak{gl}(V_{\mathcal{O}})$ be the representation carried by a registered observable family. If a Lie subspace $\mathfrak{h} \subseteq \operatorname{Ker} \rho_{\mathcal{O}*}$ contains the curvature values,*

$$F_{\text{phys}} \in \Omega^2(\mathcal{M}_\alpha, \mathfrak{h}), \tag{74}$$

then

$$\rho_{\mathcal{O}*}(F_{\text{phys}}) = 0. \tag{75}$$

This is the precise form of entropy-only blindness. Local entropies, capacities, and reconstruction errors can be matched while a curvature lives entirely in a relational sector invisible to their representation.

Relational observable completion

Let

$$I_{\text{rel}} = (I(A:B), L_{\text{overlap}}, r_{\text{red}}, C_{\text{glue}}, \ldots) \tag{76}$$

collect mutual-information, overlap-loss, redundancy-rank, and gluing-cost observables. If the induced representation is faithful on the target curvature subspace, or if its local sensitivity Jacobian has full rank there, then the curvature is locally identifiable within the registered model class. H2 does not claim global uniqueness from local rank alone.

FINITE-COST NON-ABSORPTION AUDIT

Why unrestricted lifts are inadmissible

A broad class of finite path dependences can be embedded into a sufficiently enlarged state or external record. Therefore an unrestricted state lift or boundary enlargement would make the audit vacuous. FDS requires finite capacity and finite accounting boundaries [1].

Admissible lifts

Let $\mathfrak{L}(C_{\max})$ be the set of state or boundary lifts satisfying

$$C(L) \le C_{\max}, \tag{77}$$

together with registered locality, update-window, and accessibility constraints.

Definition 7 (Cost-constrained residual obstruction)**.** *For a raw obstruction estimator $\mathcal{O}$, define*

$$\mathcal{O}_{\text{phys}}(C_{\max}) = \inf_{L \in \mathfrak{L}(C_{\max})} \|\mathcal{O}_{\text{lifted}}(L)\|. \tag{78}$$

The extended-real convention is

$$\inf \varnothing = +\infty. \tag{79}$$

Thus an empty admissible-lift set means that no registered finite lift is available within the declared budget; it does not count as successful absorption. A complexity-penalized version is

$$\mathcal{J}(L) = \|\mathcal{O}_{\mathrm{lifted}}(L)\| + \lambda C(L) + \mu \dim L. \tag{80}$$

A residual is called non-absorbable at budget $C_{\max}$ and tolerance δ only if

$$\mathcal{O}_{\mathrm{phys}}(C_{\max}) > \delta. \tag{81}$$

This is boundary-relative, not metaphysically absolute.

Ordered audit

The registered audit is

$$\begin{aligned} \mathfrak{O}_{\mathrm{raw}} &\xrightarrow{\text{descent}} \mathcal{O}_1 \xrightarrow{\text{quotient}} \mathcal{O}_2 \\ &\xrightarrow{\text{state lift}} \mathcal{O}_3 \xrightarrow{\text{boundary lift}} \mathcal{O}_4 \\ &\xrightarrow{\text{tolerance limit}} \mathfrak{O}_{\mathrm{phys}}. \end{aligned} \tag{82}$$

The order must be reported because gauge quotient, state lift, and boundary enlargement need not commute.

FINITE-RESOLUTION AND STATISTICAL TESTING

Recovery curvature estimator shared by both branches

For a small contractible loop γ of area A_γ, define

$$\widehat{F}(\gamma) = \frac{1}{A_\gamma} \log \mathrm{Hol}(\gamma), \tag{83}$$

when the holonomy remains in a registered logarithm chart. The logarithm branch, base point, and gauge alignment are fixed in advance. Error reporting uses an Ad-invariant norm, a common registered base frame, or conjugacy invariants.

Covector-branch estimator

For a held-out surface Σ, define

$$\widehat{\epsilon}_\Sigma = \left| \widehat{\oint_{\partial\Sigma} \omega_{\mathrm{th}}} - \widehat{\int_\Sigma \ell(F_{\mathrm{phys}})} \right|. \tag{84}$$

The uncertainty budget includes response-line integration, recovery-curvature estimation, surface quadrature, calibration of ℓ, units, and orientation. A multiscale family of surfaces is required to separate local discretization error from global cohomological failure.

Bundle-branch estimator

For calibration loops γ_m, estimate J_0 by minimizing

$$\sum_{m \in \mathcal{C}_{\mathrm{cal}}} \|J_0 H_{\mathrm{rec}}(\gamma_m) - H_{\mathrm{resp}}(\gamma_m) J_0\|^2 + \lambda_J \mathcal{R}(J_0), \tag{85}$$

then evaluate the frozen J_0 on $\mathcal{C}_{\mathrm{val}}$. Report validation error, singular values of the intertwiner design operator, and the dimension of the near-null intertwiner space.

Bias–variance tradeoff

A smooth-loop expansion gives geometric bias of order $O(\ell)$ after division by area for loop scale ℓ, while holonomy noise is amplified as A_γ^{-2}. Neither branch permits arbitrarily small loops or surfaces without a registered multiscale stability analysis.

Branch-specific finite-resolution bounds

A covector-branch model may report a bound of the form

$$\widehat{\epsilon}_\Sigma \le C_{\mathrm{geom}}\ell + C_{\mathrm{rec}}\varepsilon_{\mathrm{rec}} + C_{\mathrm{line}}\varepsilon_{\mathrm{line}} + C_{\mathrm{surf}}\varepsilon_{\mathrm{surf}} + C_\ell \varepsilon_\ell. \tag{86}$$

A bundle-branch model may report

$$\begin{aligned} \left\| \widehat{F}_{\mathrm{resp}} J - J \widehat{F_{\mathrm{phys}}} \right\| &\le C_{\mathrm{geom}}\ell + C_{\mathrm{hol}} \frac{\varepsilon_{\mathrm{hol}}}{\ell^2} + \left\| \nabla^{\mathrm{Hom}} B \right\| \\ &\quad + C_{\mathrm{desc}}\varepsilon_{\mathrm{desc}} + C_{\mathrm{stat}}\varepsilon_{\mathrm{stat}}. \end{aligned} \tag{87}$$

The constants and norms are model-specific; neither inequality is asserted as a device-independent law.

ANALYTIC MODELS

Rank-one Kato curvature: a scalar-visible Abelian sector

Let $\mathcal{M} = S^2$ with coordinates (θ, ϕ), and define

$$|\psi(\theta, \phi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi} \sin\frac{\theta}{2}|1\rangle. \tag{88}$$

The rank-one projector $P = |\psi\rangle\langle\psi|$ defines a Kato line bundle. In a local frame the Abelian connection is

$$A_K = i\langle\psi|\mathrm{d}\psi\rangle = -\sin^2\frac{\theta}{2}\,\mathrm{d}\phi \tag{89}$$

up to a sign convention, with curvature

$$F_K = \mathrm{d}A_K = -\frac{1}{2}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{90}$$

This is a genuine Abelian curvature that a scalar line response may detect. It is the correct scalar-visible replacement for an invalid attempt to map a pure $su(2)$ commutator to a one-dimensional response.

Pure commutator curvature: scalar-invisible, matrix-visible

On a trivial rank-two bundle over $\mathbb{R}^2$, choose

$$\Gamma_x = ia\sigma_x, \qquad \Gamma_y = ib\sigma_y. \tag{91}$$

Then

$$F_{xy} = [\Gamma_x, \Gamma_y] = -2iab\sigma_z. \tag{92}$$

Every one-dimensional representation of $SU(2)$ has zero differential, so a scalar associated response is blind to Eq. (92). The defining two-dimensional representation sees it directly. This model separates scalar blindness from physical triviality.

Central plus semisimple curvature

Let

$$\mathfrak{g}_{\text{phys}} = u(1) \oplus su(2), \qquad F_{\text{phys}} = f_0 I + f_a\sigma_a. \tag{93}$$

A scalar character can read f_0 but annihilates the $su(2)$ component. A matrix response can read both. This is the minimal algebraic model for simultaneous scalar and relational response channels.

Flat global monodromy

On a punctured plane take

$$\Gamma = \alpha T\, \mathrm{d}\phi, \tag{94}$$

with constant generator T. Locally $F = 0$, while

$$\mathrm{Hol}(\gamma) = e^{2\pi\alpha T} \tag{95}$$

for a loop around the puncture. This may label a global recovery sector but does not produce local response curvature.

Mixed compatibility obstruction

Take two regional charts with constant local connections but a parameter-dependent overlap map $g_{12}(x)$. If

$$K_{x,12} = \partial_x g_{12} + \Gamma_x^{(2)} g_{12} - g_{12}\Gamma_x^{(1)} \neq 0, \tag{96}$$

then each chart has a valid local connection but no global descended connection. A local curvature comparison before resolving $K_{x,12}$ is not a physical global recovery curvature.

THREE-QUTRIT-CODE-SEEDED CANDIDATE-SUPPORT AUDIT

Seed code and exact recovery property

Consider the three-qutrit erasure code with logical basis

$$|0_L\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle), \tag{97}$$

$$|1_L\rangle = \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle), \tag{98}$$

$$|2_L\rangle = \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle). \tag{99}$$

The undeformed code corrects erasure of any one physical qutrit, so AB, BC, and CA provide overlapping recovery sectors [20, 33, 34].

Why arbitrary support deformation is insufficient

Let P_0 project onto the code subspace and let

$$P(\theta, \phi) = U(\theta, \phi) P_0 U(\theta, \phi)^\dagger. \tag{100}$$

Constant rank guarantees a smooth support bundle but does not preserve erasure correctability [22, 26]. For each erased region R, the deformed code must satisfy

$$P E_\alpha^{(R)\dagger} E_\beta^{(R)} P = c_{\alpha\beta}^{(R)} P. \tag{101}$$

An arbitrary unitary mixing the logical support with an auxiliary subspace generally violates this condition. Such a family is a code-seeded candidate-support model, not a registered recoverable support and not a completed QEC recovery realization.

Correctability-preserving control branch

A safe branch is a product-local orbit

$$U = U_A \otimes U_B \otimes U_C. \tag{102}$$

Conjugation maps the full error algebra on an erased share to itself, so the Knill–Laflamme conditions remain valid. The decoders are conjugated versions of the undeformed decoders. This gives a genuine correctability-preserving control chamber, but it does not guarantee nonzero physical support curvature.

Support-curvature branch and evidence boundary

A support-mixing family can generate nonzero Kato support curvature. Before it is called a QEC recovery model, Eq. (101) must be verified for all three erased regions and explicit decoders supplied. H2 therefore labels this branch a *three-qutrit-code-seeded candidate-support control*. It is not evidence that nonzero support curvature coexists with full pair-region recoverability.

Companion diagnostic and release status

A deterministic companion script has been prepared for this release bundle. It constructs the seed code, evaluates Knill–Laflamme residuals, checks a correctability-preserving local-unitary control, constructs a fixed support-mixing deformation, and separately reports support curvature and KL violation. At the manuscript date no public repository DOI, release tag, or commit hash has yet been assigned; the script is therefore a packaged release companion rather than a permanently archived external resource.

The release-bundle entry point is:

`FDS_H2_qutrit_support_audit.py`

with Python 3 and NumPy as its only runtime requirements. The calculation is deterministic and uses no random seed. The auxiliary subspace is obtained by deterministic Gram–Schmidt orthogonalization of the computational basis against the code isometry. The registered support-mixing blocks are

$$B_1 = I_3, \qquad B_2 = i\ \mathrm{diag}(0.5, -0.3, 0.8), \tag{103}$$

and the finite deformation uses $(\theta, \phi) = (0.23, -0.17)$. The product-local control uses the first two Gell–Mann generators on different physical shares with the same parameters. The expected JSON output is

`FDS_H2_qutrit_support_audit_report.json`.

A registered run gives Table IV.

The result is intentionally support-level and diagnostic. The correctability-preserving branch has vanishing raw support curvature in this registered control. The nonzero-curvature branch fails the KL audit and is therefore ineligible for construction of a task-relative physical quotient. No row in Table IV establishes $F_{\mathrm{phys}} \neq 0$ or a response-visible bridge.

Open problem: curved correctable code manifolds

Criterion 3 (H2–QEC target)**.** *Determine whether there exists a smooth constant-rank family $X \mapsto P(X)$ such that [21, 24]:*

1. *erasure of each one-qutrit share remains correctable for every X;*
2. *the Knill–Laflamme conditions hold throughout the chamber;*
3. *the physical quotient of the Kato support connection has $F_{\mathrm{phys}} \neq 0$; and*
4. *at least one pre-registered response representation sees that curvature.*

An existence result would provide a genuine nontrivial H2/QEC realization. A no-go result could instead show, in a registered code class, that complete overlapping erasure correctability forces flat physical support geometry. Both outcomes are scientifically useful.

TENSOR-NETWORK AND RECOVERY-DATA PROTOCOLS

Both branches begin by fixing a finite tensor-network or stabilizer-code family, a task-relative recoverable support, a registered connection rule, descent data, mixed compatibility, and a regular physical quotient. They diverge only on the response side.

Protocol 3 (Matched-control covector test)**.** *1. Register the operational one-form ω_{th}, functional ℓ, units, normalization, and admissible surfaces.*

2. *Match local entropies, capacities, region sizes, and local recovery errors across controls.*
3. *Use calibration surfaces only to determine pre-declared constants in ℓ.*
4. *Freeze the bridge and evaluate Eq. (45) on held-out surfaces.*
5. *Test local convergence and the global de Rham obstruction separately.*
6. *Perform finite-cost memory and boundary lifts and report the surviving residual.*

Protocol 4 (Matched-control bundle test)**.** *1. Register the response bundle, connection, loop family, and candidate representation.*

2. *Match local scalar observables across controls and split loops into calibration and validation sets.*
3. *Fit J_0 only on calibration loops and freeze all choices.*
4. *Test central, semisimple, projected-null, gauge, global-flat, and rank-changing branches.*
5. *Report held-out holonomy-intertwiner error, Hom-bundle defect, loop-scale convergence, and response visibility.*
6. *State whether the tested loops generate the holonomy group or only sample a subgroup.*

TABLE IV. Three-qutrit diagnostic audit. KL entries are the maximum Frobenius residual over the three erased shares. The curvature column reports raw Kato support curvature only; no physical quotient or response representation has been constructed for these branches.

Branch	max KL residual	raw/support curvature $\|F_K^{(L)}\|_F$	physical-quotient status
Seed code	9.7×10^{-17}	–	not constructed
Product-local control	1.9×10^{-16}	0	not constructed
Support mixing	1.47×10^{-1}	1.98	ineligible: KL failure

The decisive comparison in either branch is whether systems matched in registered local scalar data but differing in physical recovery curvature separate only in the response channels predicted by the frozen bridge.

NONINVERTIBLE QUANTUM-CHANNEL SUPPORT

Constant-rank Choi support

Let $\mathcal{R}_X$ be a smooth family of completely positive maps with Choi operators $C_{\mathcal{R}}(X)$. The maps need not be invertible. Assume only that

$$\operatorname{rank} C_{\mathcal{R}}(X) = r \tag{104}$$

is constant on a chamber, and let $P_{\mathcal{R}}(X)$ project onto the Choi support.

Theorem 6 (Constant-rank channel-support connection). *The support family* $\operatorname{Im} P_{\mathcal{R}}(X)$ *defines a smooth vector bundle with Kato connection*

$$\nabla^{\text{Choi}} = P_{\mathcal{R}} \mathrm{d} \tag{105}$$

(where d *is the registered flat connection on the trivial Hilbert–Schmidt ambient bundle, consistent with the Kato canonicality discipline relative to registered ambient geometry) and curvature*

$$F_{\text{Choi}} = P_{\mathcal{R}}(\mathrm{d}P_{\mathcal{R}} \wedge \mathrm{d}P_{\mathcal{R}})P_{\mathcal{R}}. \tag{106}$$

The construction depends only on the Choi support and is invariant under changes of Kraus representation.

Proof. The smooth constant-rank support theorem and Kato construction apply to the Choi operators viewed in Hilbert–Schmidt space. Kraus changes leave the Choi operator, hence its support projector, unchanged. □

This theorem does not define an inverse transport for the full CPTP semigroup. It supplies a canonical recoverable-support geometry for a physically important noninvertible model class. By Proposition 3, if two channel families share the same Choi-support projector, this branch cannot distinguish their internal spectra or channel action.

PREMETRIC HANDOFF TO G1

Embedding physical screen data

Let

$$\iota : (p, k, \Sigma, \text{probe data}) \longmapsto X \in \mathcal{M}_{\text{scr}} \tag{107}$$

embed a registered physical screen configuration into the H2 parameter manifold. The covector branch supplies

$$\iota^* \mathrm{d}\omega_{\text{th}}, \tag{108}$$

whereas the bundle branch supplies

$$\iota^* F_{\text{resp}}. \tag{109}$$

H2 does not identify these objects without an additional registered equivalence.

Criterion 4 (Premetric G1 handoff). *An H2 response obstruction is eligible for the G1 obstruction layer only if it satisfies:*

1. *refinement naturality;*
2. *observer covariance;*
3. *finite locality;*
4. *tensorial or null-direction descent;*
5. *Ward admissibility;*
6. *no double counting with matter, memory, or external-ledger sectors;*
7. *finite-description identifiability; and*
8. *dimensional and normalization consistency.*

In addition, the covector branch must pass its functional, Stokes, and de Rham tests, whereas the bundle branch must pass its morphism, holonomy, and representation tests.

H2 states a formal gate, not a proof that these conditions hold universally.

What H2 supplies and what remains downstream

H2 supplies a registered support geometry, a regular physical quotient when the stated assumptions hold, and one of two branch-specific response-obstruction criteria. It does not supply an area-law coefficient, Einstein calibration, a conserved residual tensor, a Weyl/Ricci decomposition, the optical 3/4 coefficient, a rank-one horizon memory kernel, or a cosmological likelihood result.

BOUNDARY BETWEEN H2 AND H3

H2 may hand H3 either an operational-covector curl $d\omega_{\mathrm{th}}$ that passed the covector bridge criteria or a response-bundle curvature F_{resp} that passed the bundle bridge criteria. H3 must state which object is decomposed into the optical sectors

$$A^{(1)} \oplus S^{(2)} \oplus \Omega^{(1)}. \tag{110}$$

The two inputs are not identified automatically. In either case, the implication

$$d\omega_{\mathrm{th}} \neq 0 \quad \Longrightarrow \quad \kappa = \frac{3}{4} \tag{111}$$

is forbidden in H2.

SUCCESS CRITERIA AND DEMOTION PATHS

Minimal mathematical success

H2 reaches minimal mathematical success if it provides a valid descent hierarchy, a registered support connection, a regular physical quotient branch, the covector functional and global-obstruction criteria, the bundle holonomy-intertwiner criterion, the scalar visibility theorems, and an explicit account of the full-channel blind sector.

Covector-branch operational success

A nontrivial covector bridge realization requires:

1. $F_{\mathrm{phys}} \neq 0$ and $\ell(F_{\mathrm{phys}}) \neq 0$;
2. independently registered ω_{th}, ℓ, units, and normalization;
3. held-out Stokes tests passing at the registered tolerance;
4. finite-cost memory and leakage audits;
5. vanishing of $[\ell(F_{\mathrm{phys}})]_{\mathrm{dR}}$ whenever one global ω_{th} is claimed; and
6. successful premetric G1 handoff.

Bundle-branch operational success

A nontrivial bundle bridge realization requires:

1. $F_{\mathrm{phys}} \neq 0$ and an independently registered response bundle;
2. a nonzero, identifiable holonomy intertwiner space;
3. a frozen J_0 that passes held-out-loop validation;
4. nonzero response visibility in the registered representation;
5. finite-cost and loop-scale stability; and
6. successful premetric G1 handoff.

Demotion paths

No stable support or registered rule demotes connection identifiability. An irregular quotient demotes the ordinary physical-bundle branch. In the covector branch, failure of Stokes validation, scale calibration, parallelity of ℓ, or the global de Rham condition demotes the proposed functional bridge. In the bundle branch, failure of holonomy intertwining demotes the proposed morphism or response representation. A scalar attempt to read pure commutator curvature demotes the scalar model, not the curvature. An obstruction removed by an admissible finite-cost lift is classified as memory or leakage. Failure of the premetric handoff forbids a G1 source claim.

Non-propagation

None of these failures automatically falsifies H1, standard holography, quantum error correction, G1 as an independent response framework, general relativity, or the FDS formal core.

CONCLUSION

H2 separates regional descent, mixed compatibility, recoverable-support curvature, physical gauge reduction, and response visibility. Its principal geometric construction is support-level: relative Kato transport detects how a constant-rank support moves inside registered ambient Hermitian geometry, but it does not detect spectral weights, recovery fidelity, or channel action internal to a fixed support.

The response side has two independent branches. In the operational-covector branch, the physical input is the directly registered one-form ω_{th}. A candidate bridge is $d\omega_{\mathrm{th}} = \ell(F_{\mathrm{phys}})$, tested by held-out Stokes comparisons and constrained globally by $[\ell(F_{\mathrm{phys}})]_{\mathrm{dR}} = 0$ when a single global covector is claimed. No response bundle or holonomy intertwiner is required. In the response-bundle branch, the physical input is an independently registered response connection and curvature. A parallel bridge exists exactly when a base-point map intertwines the registered recovery and response holonomies; curvature naturality and the Hom-bundle defect follow within that branch.

The visibility analysis is stratified. Character-based one-dimensional responses see only the Lie-algebra Abelianization of the identity component; discrete characters may additionally detect global flat monodromy. General parallel readouts in the covector branch are constrained first by the actual registered holonomy group and need not factor through the full G_{phys}-Abelianization. The three-qutrit audit remains a candidate-support diagnostic rather than a completed bridge: the correctability-preserving control is flat, while the tested nonzero-support-curvature deformation fails Knill–Laflamme conditions and never reaches physical-quotient or response validation.

H2 therefore establishes a formal recoverable-support bridge architecture, branch-specific existence criteria, visibility limits, and validation protocols. A completed physical bridge would additionally require a model with nonzero physical quotient curvature and successful independent validation in at least one branch. H2's full-channel recovery branch remains open, despite related geometric constructions on quantum-channel spaces [25]. Curved correctable code manifolds, optical selection, Ward completion, and cosmological consequences remain open.

The compressed thesis is:

> Recoverable-support geometry can become a candidate premetric response obstruction through either a directly registered covector functional bridge or an independently registered response-bundle morphism, but the two routes have different local data, global obstructions, and operational tests.

Local Proof of the Holonomy-Intertwiner Theorem

Let γ_X be a locally smooth family of paths from X_0 to X. Equation (58) defines a smooth local map. For a tangent vector v at X, append a short segment δ_v to γ_X. Parallel transport along the appended segment gives

$$J_{X+\delta_v} = U_{\mathrm{resp}}(\delta_v) J_X U_{\mathrm{rec}}(\delta_v)^{-1}. \tag{112}$$

Expanding to first order yields

$$\nabla_v^{\mathrm{resp}} J - J \nabla_v^{\mathrm{rec}} = 0. \tag{113}$$

Path independence was proved in the main text using the closed-loop intertwining condition. Conversely, a parallel J intertwines all path transports by uniqueness of parallel solutions, hence all based holonomies.

Curvature of the Relative Kato Connection

Let $E = \operatorname{Im} P \subset \mathcal{H}_{\mathrm{amb}}$ and $Ps = s$. With $\nabla^K = P\nabla^{\mathrm{amb}}$,

$$(\nabla^K)^2 s = P\nabla^{\mathrm{amb}}(P\nabla^{\mathrm{amb}} s) \tag{114}$$
$$= P(\nabla^{\mathrm{amb}} P) \wedge \nabla^{\mathrm{amb}} s + P F_{\mathrm{amb}} s. \tag{115}$$

Using $P(\nabla^{\mathrm{amb}} P)P = 0$ gives

$$(\nabla^K)^2 s = P F_{\mathrm{amb}} P s + P(\nabla^{\mathrm{amb}} P \wedge \nabla^{\mathrm{amb}} P) P s, \tag{116}$$

which proves Eq. (27). The flat formula follows from $\nabla^{\mathrm{amb}} = \mathrm{d}$ and $F_{\mathrm{amb}} = 0$.

Holonomy-Invariant Readouts and Full-Group Abelianization

Holonomy-invariant parallel readouts

A parallel Branch-A readout satisfies $\nabla^{\mathrm{Hom}} \ell = 0$. At a base point X_0 this reduces to $\mathrm{Ad}_{\mathrm{Hol}}$-invariance, which implies

$$\ell_{X_0}\big([\mathfrak{hol}_{X_0}, \mathfrak{g}_{\mathrm{phys}}]\big) = 0. \tag{117}$$

The closure of the readout curvature follows from parallelity and the Bianchi identity alone, without any full-group invariance condition:

$$\mathrm{d}\,\ell(F_{\mathrm{phys}}) = (\nabla^{\mathrm{Hom}} \ell)(F_{\mathrm{phys}}) + \ell(D_{\mathrm{phys}} F_{\mathrm{phys}}) = 0. \tag{118}$$

On a contractible chamber, the componentwise Poincaré lemma then gives a local V_{cov}-valued one-form ω_{th} satisfying $\mathrm{d}\omega_{\mathrm{th}} = \ell(F_{\mathrm{phys}})$. This local existence does not identify ω_{th} with the independently measured G1 response; that equality remains an empirical bridge condition.

Full-group invariant functionals and characters

An $\mathrm{Ad}(G_{\mathrm{phys}})$-invariant linear functional $\ell : \mathfrak{g}_{\mathrm{phys}} \to \mathbb{R}$ satisfies

$$\ell([A, B]) = 0 \qquad \forall A, B \in \mathfrak{g}_{\mathrm{phys}} \tag{119}$$

by differentiating $\ell(\mathrm{Ad}_{e^{tA}} B) = \ell(B)$ at $t = 0$. Such a functional factors through the full Lie-algebra Abelianization $\mathfrak{g}_{\mathrm{phys}}^{\mathrm{ab}} = \mathfrak{g}_{\mathrm{phys}}/[\mathfrak{g}_{\mathrm{phys}}, \mathfrak{g}_{\mathrm{phys}}]$. A smooth one-dimensional group character $\rho : G_{\mathrm{phys}} \to \mathbb{C}^\times$ or $U(1)$ yields a differential ρ_* of this type. However, an

$\mathrm{Ad}(G_{\mathrm{phys}})$-invariant Lie-algebra functional need not integrate to a global group character; the group topology and period lattice may impose additional obstructions not visible at the Lie-algebra level.

Regular Quotient Conditions

The regular quotient theorem assumes:

1. G_{tr} and G_0 are finite-dimensional Lie groups in the registered model;
2. G_0 is closed and normal;
3. the right G_0-action on P_{rec} is free and proper;
4. orbit type is constant on the recovery chamber;
5. the original connection is a genuine G_{tr}-connection.

If any condition fails, H2 does not infer a smooth physical vector bundle. The appropriate replacement may be a stratified quotient, orbifold, Lie groupoid, or quotient stack.

Near-Cocycle Stability and Liftability

A universal near-cocycle correction theorem is not claimed. Quotient descent and strict liftability through Eq. (15) are different questions. In a finite good cover with compact matrix groups, a fixed logarithm chart, a registered trivial lifting obstruction for the extension in Eq. (11), and sufficiently small defect, one may seek a G_0-valued one-cochain that improves the lifted cocycle. The constants depend on the cover nerve, group norm, chart radius, twisting action, and lifting class. A small quotient defect does not by itself imply the existence of a nearby strict G_{tr} lift.

Finite-Code Support Curvature and Correctability Audit

For a general support-mixing family,

$$P_0 = \begin{pmatrix} I_L & 0 \\ 0 & 0 \end{pmatrix}, \qquad G_\mu = \begin{pmatrix} 0 & B_\mu^\dagger \\ B_\mu & 0 \end{pmatrix}. \tag{120}$$

In a flat registered ambient bundle, the Kato curvature on the logical block at the origin is

$$P_0[\partial_\theta P, \partial_\phi P]P_0 = B_1^\dagger B_2 - B_2^\dagger B_1 \tag{121}$$

up to sign convention. This shows that support curvature can be engineered; it does not show that the support remains an erasure-correcting code. That stronger claim requires Eq. (101) for every erased region. The companion script evaluates both quantities separately.

Support Geometry versus Full-Channel Geometry

The Kato branch factors through the map

$$\mathcal{R}(X) \longmapsto P_{\mathcal{R}}(X). \tag{122}$$

Its kernel contains every variation of the recovery channel that preserves the registered support projector. Consequently, spectral redistribution, changes of positive Choi weights, and changes of channel action within a fixed support are invisible to ∇^K and F_K. A future full-channel branch must enrich the geometric object beyond the projector, for example through a metric and connection on positive Choi operators or an operationally selected family of recovery maps. H2 does not choose among those possibilities.

FDS Internal Crosswalk

Within the FDS twelve-component object, H2 uses:

State X:
the registered screen-state and recovery-control coordinates;

Environment E:
inaccessible bulk, bath, decoder, or external-record degrees of freedom;

Boundary B:
the finite causal screen and its cover;

Memory M:
local recovery frames, decoder state, and admissible finite lifts;

Observation Y:
accessible screen and response data;

Action A:
recovery, transport, gauge fixing, and probe operations;

Update U:
regional and parameter-space recovery transport;

Projection π:
finite task-relative recovery support and physical quotient;

Loss ℓ:
recovery, overlap, bridge, and validation loss;

Budget Φ:
capacity, complexity, state-lift, and boundary-lift budgets;

Perturbations $\mathcal{P}$:
registered screen, channel, and code deformations;

Window τ:
recovery, calibration, validation, and response timescale.

Notation Summary

$A_a(X)$**:**
regional screen patch at state X.

$\mathcal{Q}_a(X)$**:**
task-relative local recoverable sector.

g_{ab}**:**
regional recovery intertwiner.

h_{abc}**:**
regional descent or lifting defect.

$K_{i,ab}$**:**
mixed recovery–gluing defect.

$P(X)$**:**
registered recoverable-support projector.

Γ_a, F_a**:**
regional local connection and curvature.

$\Gamma_{\mathrm{tr}}, F_{\mathrm{tr}}$**:**
connection and curvature on the principal transport bundle before physical quotient.

G_0**:**
task-invariant recovery gauge subgroup.

$P_{\mathrm{rec}}^{\mathrm{phys}}, G_{\mathrm{phys}}$**:**
physical principal recovery bundle and structure group.

$\Gamma_{\mathrm{phys}}, F_{\mathrm{phys}}$**:**
physical quotient connection and curvature.

$P_{\mathrm{resp}}, A_{\mathrm{resp}}, F_{\mathrm{resp}}$**:**
independently registered response principal bundle, connection, and curvature.

s**:** physically registered local response frame, when the response-frame criterion holds.

ω_{th}**:**
independently registered operational G1 response covector in the covector branch.

ω_{loc}**:**
registered local response potential in the response-bundle branch, when the response-frame criterion holds.

J, J_0**:**
parallel recovery–response morphism and its base-point value.

B**:** Hom-bundle bridge defect $\nabla^{\mathrm{Hom}} J$.

ℓ**:** pre-registered parallel flat-coefficient readout, scalar or finite multicomponent; full-group invariance is an additional condition.

$\mathfrak{O}_{\mathrm{phys}}(C_{\max})$**:**
cost-constrained residual obstruction.

CODE AND RELEASE-PACKAGE AVAILABILITY

The release package contains the deterministic script `FDS_H2_qutrit_support_audit.py`, the expected report `FDS_H2_qutrit_support_audit_report.json`, `requirements.txt`, a reproduction README, and `SHA256SUMS.txt`. The registered command is

```
python FDS_H2_qutrit_support_audit.py.
```

The calculation is deterministic and uses no random seed. At the manuscript date no public repository, Zenodo DOI, release tag, or commit hash has been assigned. The files are therefore release-package companion materials rather than a permanently archived external repository. A public archival release must replace this paragraph with the repository URL, DOI, release tag, commit hash, exact environment, and output checksums.

AI ASSISTANCE DISCLOSURE

This manuscript draft may have used AI-assisted mathematical organization, prose editing, LaTeX preparation, and consistency checking under the author's direction. The conceptual claims, theorem assumptions, interpretation boundaries, numerical implementation, and scientific responsibility remain with the author.

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