Boundary Maintenance and the Second Law under Finite Memory: Irreversible Record Reuse, Entropy Ledgers, and Operational Time Arrows

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FDS-O3 develops the third paper in the Operational Trident. O1 treated observation as finite stable record formation, O2 treated register time as causally ordered irreversible update, and O3 treats the Second Law as an operational channel instantiated by finite-memory boundary maintenance. The paper does not rederive the Second Law from distinction alone. It does not claim that mathematical coarse-graining is heat, that every update dissipates $k_{\mathrm{B}}T\ln 2$, or that topological invariants violate thermodynamics. Its narrower claim is conditional: in a physically realized active finite distinction system, finite memory turns sustained boundary maintenance into record reuse; record reuse creates residual irreversibility unless inverse information remains available in side records; and physical overwrite, cleanup, refresh, repair, synchronization, or externalization must be accounted for in a coupled entropy or resource ledger. The central theorem states that sustained residual record turnover, fixed boundary-maintenance tolerance, and zero coupled-system entropy/resource cost cannot persist indefinitely under finite memory and physical bridge assumptions. Deterministic simulations illustrate memory-fill pressure, residual irreversibility, finite-record entropy ledgers, externalization audits, pruning return on investment, invariant compression, overload hysteresis, and finite-memory regime diagrams. O3 thereby supplies a finite-record operational channel connecting FDS-P1/P2/P5 to macroscopic irreversibility without claiming to replace statistical mechanics or stochastic thermodynamics.

Scope and Boundary of the Theory. This paper is an operational physical-bridge paper. It does not derive the Second Law from the FDS formal core alone. It does not claim that every formal distinction, observation, or coarse-graining dissipates heat. It does not claim that all thermodynamic irreversibility is Landauer erasure, nor that finite memory is the only foundation of thermodynamics. It claims only that physically realized active finite systems with bounded memory instantiate an operational Second-Law channel when they maintain a boundary through irreversible record reuse, refresh, repair, synchronization, externalization, or recovery under finite resource budgets. FDS-O3 does not identify thermodynamic entropy with memory use. It studies *record-turnover entropy*: the entropy/resource cost incurred when finite physical records are reused, erased, refreshed, repaired, synchronized, or externalized.

Claim-status summary

Table I summarizes the main claims and failure conditions.

Keywords: finite memory; Second Law; record reuse; residual irreversibility; boundary maintenance; active finite distinction systems; entropy ledger; Landauer principle; housekeeping dissipation; register time; pruning; externalization; invariant compression; stochastic thermodynamics.

INTRODUCTION

Why finite memory matters for the Second Law

The classical Second Law is usually stated at the level of heat, work, and entropy. FDS-O3 asks a narrower operational question: what does macroscopic irreversibility look like from the standpoint of a finite boundary-maintaining system whose usable past exists only through finite records? A finite record-bearing system can update only by writing, preserving, compressing, externalizing, reusing, or destroying record carriers. The directionality of such a system is therefore not merely a philosophical arrow of time. It is a ledger of finite record turnover.

The FDS formal core defines active finite distinction systems as finite-capacity systems that maintain a boundary through state-dependent updates under resource constraints [1]. Its core chain is

$$\begin{aligned}&\text{finite capacity} \to \text{capacity deficit} \to \text{approximation}\\ &\to \text{irreversible update} \to \text{dissipation}\\ &\to \text{prune/externalize/collapse}.\end{aligned} \tag{1}$$

O3 turns the final thermodynamic portion of this chain into a finite-memory operational statement.

From O1, O2, P1, P2, and P5 to O3

O1 treated observation as stable finite record formation [2]. O2 treated register time as causally ordered

TABLE I. Central FDS-O3 claims, epistemic status, and failure or demotion conditions.

Claim	Status	What would weaken or falsify it
Finite memory creates record-reuse pressure under sustained update unless history is externalized, compressed, uncomputed, or abandoned.	Operational theorem	A bounded-memory system maintains unbounded usable update history internally without record reuse, external memory, compression, task relaxation, or failure.
Non-injective record reuse creates residual irreversibility relative to an accounting boundary.	Formal information claim	A many-to-one update preserves full preimage information without side records or an enlarged boundary.
Physical irreversible record reuse enters an entropy/resource ledger under bridge assumptions.	Physical bridge	Reliable physical erasure or overwrite violates Landauer-style accounting under the stated assumptions.
Stable finite records require housekeeping beyond logical erasure.	Accounting claim	Refresh, retention, clocking, synchronization, carrier repair, and verification are cost-free in every controlled finite-record implementation.
Externalization shifts the operational Second-Law channel across accounting boundaries rather than eliminating it.	Accounting bridge	External records impose no write, verification, retrieval, latency, maintenance, or environmental cost.
Pruning and invariant compression can reduce future entropy pressure when task identity is preserved.	Conditional bridge	No task-preserving quotient, pruning, or compression ever reduces future record-maintenance cost.
Sustained residual record turnover, fixed boundary tolerance, and zero coupled entropy/resource cost cannot persist indefinitely.	O3 theorem	A finite active-boundary system maintains sustained residual record turnover at fixed tolerance with no ledger cost and no exit channel.
Topological or invariant persistence, if present, redirects entropy accounting rather than violating the Second Law.	Quarantined projection	A protected invariant supplies perpetual work or global entropy-law violation rather than bounded persistence or entropy relocation.

irreversible finite-record update [3]. P1 defined physical distinction carriers and accounting boundaries [5]. P2 showed why bounded-memory reversible computation accumulates garbage histories unless uncomputed, externalized, compressed, cleaned, or abandoned [6]. P5 gave a generalized entropy ledger for capacity deficit and boundary maintenance [8]. O3 combines these into a single statement:

$$\begin{aligned} &\text{finite records} + \text{bounded memory} \\ &+\text{continued boundary maintenance} \\ &\Longrightarrow \text{operational Second-Law channel.} \end{aligned} \tag{2}$$

What O3 does not claim

O3 does not replace statistical mechanics, fluctuation theorems, or stochastic thermodynamics. It does not say that finite memory is the metaphysical source of thermodynamics. It does not identify formal coarse-graining with heat. It does not use topological protection as a counterexample to the Second Law. Its claim is conditional and operational: in physically realized finite-memory systems, sustained boundary maintenance requires record turnover; if that turnover is irreversible within the accounting boundary, it appears in a coupled entropy/resource ledger or is replaced by an exit channel.

CORE FDS INGREDIENTS

Active finite distinction systems

The FDS core object is

$$S = (X, E, B, M, Y, A, U, \pi, \ell, \Phi, \mathcal{P}, \tau), \tag{3}$$

where X is internal state, E environment, B boundary, M memory or model, Y observation channel, A action space, U update rule, π finite projection, ℓ boundary-maintenance loss, Φ resource budget, $\mathcal{P}$ perturbation or pruning family, and τ the update timescale. The timescale matters because dimensionless information counts become rates only after division by τ.

An active boundary requires nontrivial update and relevance to future boundary loss:

$$\mathbb{P}\{U(M_t, Y_t) \neq M_t\} > 0, \qquad \mathrm{I}(M_{t+1}; \ell_{t+k}) > 0. \tag{4}$$

TABLE II. Relation between O3 and nearby FDS bridge papers. O3 is not a repeat of P5; it shifts the object from capacity deficit to finite-memory record turnover.

Paper	Core object	Main bridge
P1	physical carriers	distinction → carrier / erasure map
P2	reversible history	side records → finite reuse pressure
P5	capacity deficit	deficit → entropy-production ledger
O3	finite memory	record turnover → operational time arrow / operational Second-Law channel

Empirical use requires an intervention or ablation comparison between the update rule and a frozen, randomized, or identity-update null channel.

Capacity deficit and update windows

Let $Z_t = \psi(E_t, B_t)$ be a pre-registered task-relevant statistic. Let $R^{(\tau)}_{\min}(\epsilon; \Psi_t)$ be the minimum number of bits per update window τ needed to encode an admissible statistic to distortion tolerance ϵ. For effective capacity $C_{\text{eff}}(t)$,

$$\Delta_\epsilon(t) = R^{(\tau)}_{\min}(\epsilon; \Psi_t) - C_{\text{eff}}(t). \tag{5}$$

When $\Delta_\epsilon > 0$, the system cannot internally maintain all task-relevant distinctions at the required fidelity without approximation, externalization, task relaxation, or failure.

Logical erasure and physical bridge

For an update $M_{t+1} = U(M_t, Y_t)$, logical erasure is

$$b_t = H(M_t \mid M_{t+1}, Y_t). \tag{6}$$

Under Landauer-style physical assumptions, the informational entropy-production floor is

$$\dot{\Sigma}_{\text{erase}} \geq \frac{k_B \ln 2}{\tau} H(M_t \mid M_{t+1}, Y_t). \tag{7}$$

If the update is logically reversible relative to the available side information, this term can vanish. It does not follow that the system is thermodynamically cost-free, since refresh, clocking, carrier stability, synchronization, and external verification may still be required.

Free-energy budget and maintenance failure

A physically realized active finite system obeys a resource balance of the form

$$\begin{aligned} \Phi_{t+1} \leq \Phi_t &+ \int_t^{t+\tau} \dot{F}_{\text{in}}(s) ds \\ &- \int_t^{t+\tau} \dot{Q}_{\text{ledger}}(s) ds - D_\ell(\ell_t). \end{aligned} \tag{8}$$

Here $\dot{Q}_{\text{ledger}}$ is the audited resource-cost rate associated with record reuse, refresh, repair, synchronization, externalization, and recovery. It may be modeled as

$$\dot{Q}_{\text{ledger}} = \sum_{i,r} \dot{Q}_{i,r} + \dot{W}_{\text{maint}} + \dot{C}_{\text{resource}}, \tag{9}$$

where $\dot{W}_{\text{maint}}$ denotes work used for maintenance and $\dot{C}_{\text{resource}}$ denotes resource-equivalent costs not naturally written as heat flows. When a single effective temperature is meaningful, one may use the shorthand $\dot{Q}_{\text{ledger}} \approx T_{\text{eff}} \dot{\Sigma}^{\mathcal{A}}_{\text{O3}}$. The main O3 claim does not require a single-temperature reservoir. The term D_ℓ tracks functional damage or resource loss due to boundary failure. If the ledger cost exceeds input without reserves, external subsidy, task relaxation, pruning, or externalization, the same boundary-maintenance regime cannot persist indefinitely.

FINITE MEMORY AND RECORD REUSE

Definition 1 (Finite record memory). *Let M be the system's internal record state space. The finite memory capacity is*

$$C_{\text{mem}} = \log_2 |M|, \tag{10}$$

or the corresponding operational capacity under finite resolution, noise, and coding constraints.

Definition 2 (Task-relevant record load). *Let $R^{\text{task}}_{1:t}$ denote the minimal task-relevant record history required to maintain the specified boundary tolerance (ϵ, Ψ) up to time t, and let G_t be accessible side information or garbage records that remain inside the accounting boundary. Define*

$$B_R(t; \epsilon, \Psi) = H(R^{\text{task}}_{1:t} \mid G_t). \tag{11}$$

This is not the entire microscopic history of the universe or device. It is the usable finite-record history needed for the boundary-maintenance task. The memory-fill ratio is

$$\rho_M(t) = \frac{B_R(t; \epsilon, \Psi)}{C_{\text{mem}}}. \tag{12}$$

Definition 3 (Record reuse pressure). *A finite-memory system has record reuse pressure*

$$\Pi_{\text{reuse}}(t) = [B_R(t; \epsilon, \Psi) - C_{\text{mem}}]_+. \tag{13}$$

Positive reuse pressure does not by itself imply heat. It means that continued operation requires compression, externalization, uncomputation, overwrite, cleanup, task relaxation, memory expansion, or failure.

Proposition 1 (Finite-memory reuse necessity)**.** *If $B_R(t;\epsilon,\Psi)$ grows without bound while $C_{\rm mem}$ is fixed, then there exists a finite time after which the system cannot keep the full record history internally without reuse, externalization, compression, task relaxation, or failure.*

Proof. By assumption $B_R(t;\epsilon,\Psi)\to\infty$ and $C_{\rm mem}<\infty$. Therefore for some t^*, $B_R(t;\epsilon,\Psi)>C_{\rm mem}$ for all $t>t^*$. Full internal retention would require more task-available record capacity than the memory space carries. The system must reduce demand, enlarge the boundary, move records outside the internal memory, compress them, overwrite them, or fail to preserve the full history. □

RESIDUAL IRREVERSIBILITY

Definition 4 (Boundary-relative residual irreversibility)**.** *Let $G_{\mathcal{A},t}$ be accessible side records inside accounting boundary $\mathcal{A}$. The one-step residual irreversibility of an update is*

$$L_{\mathcal{A}}(t)=H(M_t\mid M_{t+1},Y_t,G_{\mathcal{A},t}). \tag{14}$$

For a history,

$$\mathcal{I}^{\rm res}_{\mathcal{A}}(T)=H(M_{0:T}\mid M_T,Y_{0:T},G_{\mathcal{A},T}). \tag{15}$$

Residual irreversibility is not a metaphysical loss of the past. It is an accounting statement: within the chosen physical boundary, there is not enough accessible side information to reconstruct the preimage history.

Definition 5 (Finite-window residual record-turnover rate)**.** *For an update window $[t,t+\tau]$, define*

$$r^{\mathcal{A}}_{\rm res}(t;\tau)=\frac{1}{\tau}H(M_t\mid M_{t+\tau},Y_{t:t+\tau},G_{\mathcal{A},t+\tau}). \tag{16}$$

Equivalently, for a backward history window one may use

$$r^{\mathcal{A}}_{\rm res}(t;\tau)=\frac{1}{\tau}H(M_{t-\tau:t}\mid M_t,Y_{t-\tau:t},G_{\mathcal{A},t}), \tag{17}$$

when that convention better matches the instrument. For path-dependent tasks where knowledge of a single preimage state is insufficient, M_t in Eq. (16) may be replaced by the relevant pre-window path segment $M_{t-\tau:t}$ or by the minimal task-relevant record path required by the boundary task. Both forms are finite-window residual turnover rates, not global thermodynamic entropy rates.

Definition 6 (Long-run residual turnover rate)**.** *For discrete update windows of width τ, define*

$$\bar{r}^{\mathcal{A}}_{\rm res}=\liminf_{n\to\infty}\frac{1}{n\tau}H(M_{0:n}\mid M_n,Y_{0:n},G_{\mathcal{A},n}). \tag{18}$$

The theorem below uses the finite-window rate on a positive-density set of update windows; the long-run rate is a compact summary when the limit is meaningful.

Remark 1. *A reversible embedding can make $L_{\mathcal{A}}(t)=0$ by preserving inverse information in side records. Under bounded memory, those side records themselves become physical carriers requiring refresh, synchronization, externalization, uncomputation, cleanup, or loss.*

THE OPERATIONAL SECOND-LAW CHANNEL

Entropy accounting boundary

Definition 7 (O3 entropy ledger)**.** *Let $\mathcal{A}$ be an accounting boundary and let $\mathcal{P}_{\mathcal{A}}$ be a pre-registered partition of physical processes inside or coupled to that boundary. Define*

$$\dot{\Sigma}^{\mathcal{A}}_{\rm O3}=\sum_{p\in\mathcal{P}_{\mathcal{A}}}\dot{\Sigma}(p), \tag{19}$$

where each physical process is counted once. Named channels are audit classes, not automatically independent mechanisms.

A class-level representation is

$$\dot{\Sigma}^{\mathcal{A}}_{\rm O3}=\dot{\Sigma}_{\rm erase}+\dot{\Sigma}_{\rm refresh}+\dot{\Sigma}_{\rm repair}+\dot{\Sigma}_{\rm sync}+\dot{\Sigma}_{\rm ext}+\dot{\Sigma}_{\rm rec}. \tag{20}$$

If a process belongs to more than one class in an implementation, the ledger must assign it once or add overlap corrections.

For a channel i coupled to reservoirs $r\in\mathcal{R}_i$, the expected or coarse-grained stochastic-thermodynamic form is

$$\langle\dot{\Sigma}_i\rangle=\langle\dot{S}^{\rm sys}_i\rangle+\sum_{r\in\mathcal{R}_i}\left\langle\frac{\dot{Q}_{i,r}}{T_r}\right\rangle\geq 0. \tag{21}$$

Instantaneous trajectory-level entropy production can fluctuate. O3 concerns the audited expected or coarse-grained coupled ledger over the specified accounting boundary. The single-temperature form $T^{-1}\dot{Q}_i$ is only a special case.

Erasure, refresh, repair, synchronization, and externalization

The erasure contribution is bounded by

$$\dot{\Sigma}_{\rm erase}\geq\frac{k_{\rm B}\ln 2}{\tau}L_{\mathcal{A}}(t). \tag{22}$$

The remaining channels cover physical maintenance: refreshing unstable carriers, repairing boundary damage, synchronizing records across subsystems, verifying external records, and recovering from loss. These are not all Landauer erasure. They are physical processes that keep finite records task-available.

Functional loss versus entropy production

A failure of the task is not automatically entropy production. It should be tracked separately by $D_\ell(\ell)$ unless recovery, repair, uncontrolled relaxation, reset, or reconstruction is physically implemented. O3 follows the P5 distinction between physical entropy production and functional boundary loss [8].

MAIN THEOREM

Theorem 1 (Finite-memory boundary-maintenance theorem)**.** *Consider a physically realized active-boundary FDS with finite memory capacity, finite update window τ, finite resource input, and fixed boundary-maintenance tolerance $\ell(t) \le \ell_c$. Suppose: (i) sustained operation generates positive finite-window residual turnover $r^{\mathcal{A}}_{\mathrm{res}}(t_j;\tau) > 0$ on a positive-density set of update windows; (ii) accessible side records, externalization, reversible uncomputation, task relaxation, invariant compression, and resource expansion are bounded; and (iii) physical cleanup, overwrite, refresh, repair, synchronization, or externalization channels have nonzero coupled-system entropy or resource cost. Then the system cannot indefinitely maintain the same boundary task at zero coupled-system entropy/resource cost. It must enter at least one channel: physical entropy production, exported entropy production, pruning, invariant compression, task relaxation, resource expansion, passive degradation, or collapse.*

Proof. Positive residual record turnover means that, relative to boundary $\mathcal{A}$, some preimage history needed for full reconstruction is not carried by the current state plus accessible side records. If the task tolerance is fixed and the system continues active maintenance, it must compensate by preserving additional side information, reversing the update path, externalizing records, compressing history, relaxing the task, repairing damage, or physically overwriting/cleaning records. By hypothesis the non-dissipative or relief channels are bounded, and physically realized cleanup, overwrite, refresh, repair, synchronization, or externalization has nonzero coupled ledger cost. If the cost remains within the finite resource envelope, the system persists by paying or exporting entropy. If it exceeds the envelope and no exit channel is available, Eq. (8) drives maintenance failure. Thus zero-cost indefinite maintenance under sustained record turnover is impossible under the stated assumptions. □

In model classes where physical record-turnover entropy is lower-bounded by an implementation-dependent coefficient $\kappa > 0$, one may write

$$\int_0^T \langle \dot{\Sigma}^{\mathcal{A}}_{\mathrm{O3}}(t)\rangle dt \ge \kappa \int_0^T [r^{\mathcal{A}}_{\mathrm{res}}(t;\tau) - \mathcal{E}_{\mathrm{exit}}(t)]_+ dt. \tag{23}$$

Here κ has units of entropy per record unit, and $\mathcal{E}_{\mathrm{exit}}(t)$ is record-equivalent relief per update window from admissible exit channels. A minimal decomposition is

$$\mathcal{E}_{\mathrm{exit}}(t) = \mathcal{E}_{\mathrm{prune}}(t)+\mathcal{E}_{\mathrm{ext}}(t)+\mathcal{E}_{\mathrm{inv}}(t)+\mathcal{E}_{\mathrm{relax}}(t)+\mathcal{E}_{\mathrm{uncomp}}(t)+\mathcal{E}_{\mathrm{res}}(t)$$

where $\mathcal{E}_{\mathrm{prune}}$ is pruning or cleanup that reduces future record burden, $\mathcal{E}_{\mathrm{ext}}$ is externalization that supplies effective record space, $\mathcal{E}_{\mathrm{inv}}$ is invariant compression that reduces maintained history, $\mathcal{E}_{\mathrm{relax}}$ is task relaxation that lowers record demand, $\mathcal{E}_{\mathrm{uncomp}}$ is reversible uncomputation that recovers inverse paths, and $\mathcal{E}_{\mathrm{res}}$ is resource, storage, or channel expansion. All terms are expressed in record-equivalent relief units over the update window. Equation (23) is a model-class bridge, not a universal identity.

Corollary 1 (Finite-memory operational Second-Law impossibility triangle)**.** *Under finite-memory and physical bridge assumptions, the following cannot persist indefinitely without an exit channel:*

$$r^{\mathcal{A}}_{\mathrm{res}}(t;\tau) > 0, \qquad \ell \le \ell_c, \qquad \langle \dot{\Sigma}^{\mathcal{A}}_{\mathrm{O3}}\rangle = 0. \tag{24}$$

At least one term must change: residual turnover is reduced, the boundary tolerance is relaxed, entropy is produced or exported, resources expand, passive degradation is accepted, or the boundary-maintenance regime fails.

Finite-memory operational Second-Law channel. Sustained residual record turnover, fixed boundary tolerance, and zero coupled entropy/resource cost are jointly impossible for physically realized finite-memory boundary-maintaining systems under the O3 bridge assumptions.

MACROSCOPIC ARROWS FROM FINITE RECORDS

Register-time arrow

O2 treated usable time for bounded observers as a causally ordered sequence of finite record updates [3]. In O3, the same ordered update becomes thermodynamic when physical record carriers are reused. A formal update sequence $M_t \to M_{t+1}$ is not heat by itself; but if a physical system must erase, reset, refresh, or repair records to continue the sequence, the update order becomes a physical ledger.

Memory-fill arrow

Without cleanup or compression, record load tends to accumulate:

$$\rho_M(t) = \frac{B_R(t;\epsilon,\Psi)}{C_{\mathrm{mem}}} \uparrow . \tag{25}$$

A reversible system can delay erasure by preserving inverse information. In bounded memory, the delayed history becomes a memory-fill and refresh burden.

Entropy-ledger arrow

Under physical bridge assumptions,

$$\left\langle \dot{\Sigma}^{\mathcal{A}}_{\mathrm{O3}} \right\rangle \geq 0 \tag{26}$$

for the audited expected or coarse-grained coupled boundary. Local entropy can decrease, local memory pressure can fall, and local records can be exported, but the coupled ledger must include the processes that make those changes possible.

Finite-register operational equilibrium

O3 does not identify thermodynamic equilibrium with a full memory register. It proposes a narrower finite-register analogue: when usable free record capacity vanishes, and no pruning, externalization, compression, erasure, repair, or resource input is available, the register can no longer form new stable O1 records or support new O2 register-time updates [2, 3]. In this operational sense,

$$\begin{aligned} &\rho_M(t) \to 1, \qquad \mathcal{E}_{\mathrm{exit}}(t) \to 0 \\ &\qquad \Longrightarrow \text{no new task-available record capacity.} \end{aligned} \tag{27}$$

This is a conjectural analogy for finite registers, not a redefinition of thermodynamic equilibrium.

Record-turnover power floor

For an erasure-dominated implementation, if B_{turn} bits are irreversibly turned over per update window, the minimum erasure-limited power obeys

$$P^{\min}_{\mathrm{erase}} \geq \eta_{\mathrm{impl}} \frac{k_{\mathrm{B}} T \ln 2}{\tau} B_{\mathrm{turn}}, \tag{28}$$

where $\eta_{\mathrm{impl}} \geq 1$ is a toy parametrization of finite-time, reliability, control, and architectural overhead. It is not a universal multiplicative law. If nearly the full memory is turned over each window, $B_{\mathrm{turn}} \approx C_{\mathrm{mem}}$. Conversely, given an available power budget P_{avail} and housekeeping load P_{hk}, the irreversible turnover rate is bounded by

$$\dot{B}^{\max}_{\mathrm{turn}} \leq \frac{P_{\mathrm{avail}} - P_{\mathrm{hk}}}{\eta_{\mathrm{impl}} k_{\mathrm{B}} T \ln 2}. \tag{29}$$

These expressions are throughput bounds for erasure-limited channels, not universal power laws for all computation.

Hysteresis after overload

If overload forces erasure, damage, or invariant destruction, reducing demand later need not restore the original state. The hysteresis gap can be decomposed as

$$\mathcal{H}_{\mathrm{O3}} = \mathcal{H}_{\mathrm{record}} + \mathcal{H}_{\mathrm{damage}} + \mathcal{H}_{\mathrm{invariant}}, \tag{30}$$

where the three terms correspond to erased records, damaged carriers or boundaries, and lost low-maintenance invariant support. A measured path gap may be written schematically as

$$\mathcal{H}_{\mathrm{O3}} = \dot{\Sigma}_{\mathrm{down}}(r_{\mathrm{res}}) - \dot{\Sigma}_{\mathrm{up}}(r_{\mathrm{res}}) > 0. \tag{31}$$

The gap is a reconstruction burden, not a universal hysteresis law. In T3 language, part of the burden can be the activation-like cost of rebuilding Phase-B invariants destroyed during overload [4].

PRUNING, EXTERNALIZATION, AND INVARIANT COMPRESSION

Pruning as entropy-relief policy

Pruning may incur a one-time cost Σ_{prune} while reducing future record-maintenance load. Over horizon H, define

$$\begin{aligned} \mathrm{ROI}^{\mathrm{prune}}_{\Sigma}(H) = \frac{1}{\Sigma_{\mathrm{prune}} + \lambda D_{\mathrm{task}}} \Bigg[\int_t^{t+H} (\dot{\Sigma}^{\text{no prune}}_{\mathrm{O3}} - \dot{\Sigma}^{\mathrm{prune}}_{\mathrm{O3}}) ds \\ - \Sigma_{\mathrm{prune}} \Bigg]. \end{aligned} \tag{32}$$

Pruning is favorable only when it preserves task function sufficiently that future ledger savings exceed pruning cost and task loss.

Externalization as boundary shift

Externalization changes the accounting boundary:

$$\mathcal{A}_{\mathrm{local}} \to \mathcal{A}_{\mathrm{coupled}} = \mathcal{A}_{\mathrm{local}} \cup \mathcal{A}_{\mathrm{ext}}. \tag{33}$$

Local ledger cost may fall, but the coupled ledger includes writing, verification, retrieval, latency, synchronization, carrier upkeep, and environmental noise. Externalization is therefore a shift of the operational Second-Law channel, not deletion of it.

Invariant-supported compression

Let $q : X \to Q$ be an invariant quotient under perturbation family $\mathcal{P}$, with $q \circ P_i = q$ for all $P_i \in \mathcal{P}$. If the

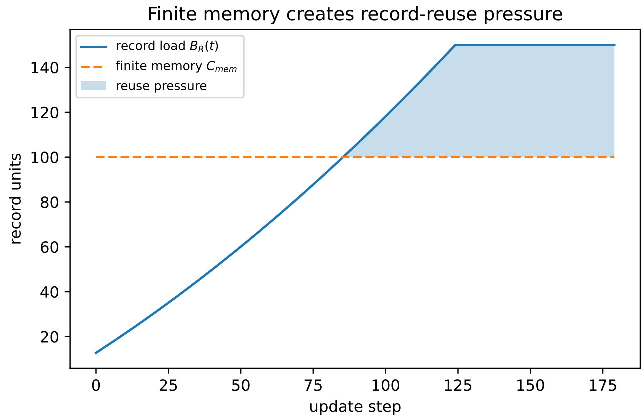

FIG. 1. Finite memory creates record-reuse pressure. Record load crosses finite memory capacity, activating reuse pressure rather than forcing a unique exit.

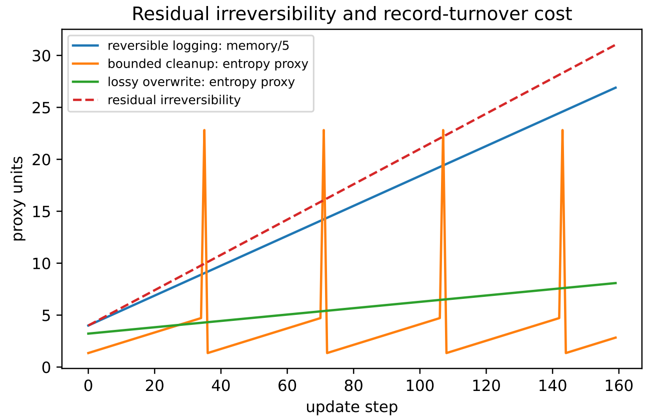

FIG. 2. Residual irreversibility and record-turnover cost. Reversible logging delays erasure by growing side records; bounded cleanup creates entropy bursts; lossy overwrite creates residual irreversibility.

task identity predicate factors as $R_A = \bar{R}_A \circ q$, then the system can maintain task identity through q instead of raw microstate detail. Invariant compression reduces the finite-memory channel when

$$\dot{\Sigma}_q + \lambda D^q_{\text{task}} < \dot{\Sigma}_X + \lambda D^X_{\text{task}}. \tag{34}$$

Compression is not monotonically beneficial. Let q index compression strength and define

$$J(q) = \dot{\Sigma}_{\text{maint}}(q) + \lambda D_{\text{task}}(q). \tag{35}$$

The stable Phase-B quotient is not the maximum compression but the minimizer $q^* = \arg\min_q J(q)$ under the task-tolerance constraint. Under-compression carries unnecessary record burden; over-compression destroys task-relevant distinctions and raises D_{task}. Topological protection, error-correcting code spaces, and robust organizational routines are domain projections of this form only when an explicit bridge maps the invariant quotient to physical record maintenance.

NUMERICAL MODELS AND SIMULATIONS

The simulations are deterministic synthetic demonstrations. They are not fits to physical devices, biological organisms, quantum experiments, or human-subject data. They illustrate the O3 normal form and are generated by `code/generate_results.py`.

EXPERIMENTAL AND ENGINEERING PROTOCOLS

Protocol 1 (Finite-memory reuse protocol). *Hold memory capacity fixed while increasing update rate, task demand, or retention horizon. Independently estimate the task-relevant record demand and memory capacity where possible; the crossing point should be estimated before observing heat, reset, latency, or failure signatures whenever possible. Measure reset rate, energy or heat, latency, error floor, buffer overflow, pruning, and externalization. O3 predicts a co-occurrence of exits when record demand crosses finite memory capacity.*

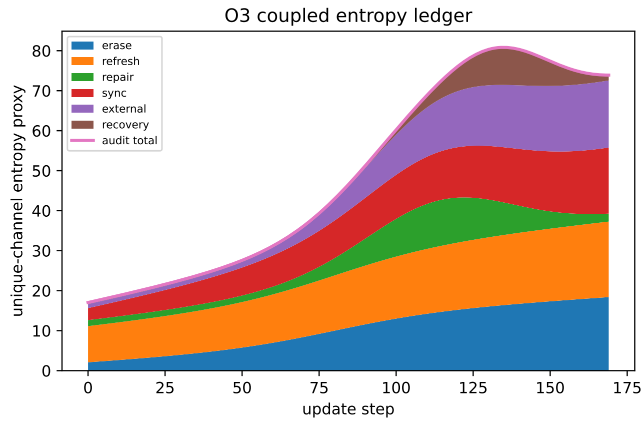

FIG. 3. O3 coupled entropy ledger. Stacked areas are illustrative unique-channel assignments under an audit convention. The audit total is not an additional channel.

Protocol 2 (Reversible logging versus cleanup). *Compare three implementations of the same task: reversible logging with expanding memory, bounded reversible logging with cleanup, and irreversible overwrite. Audit the coupled ledger including side-record refresh, synchronization, cleanup heat, and task loss. The claim is not that reversible computation fails; it is that bounded reversible history requires physical management.*

Protocol 3 (Externalization audit). *Move records outside the local boundary and measure local savings together*

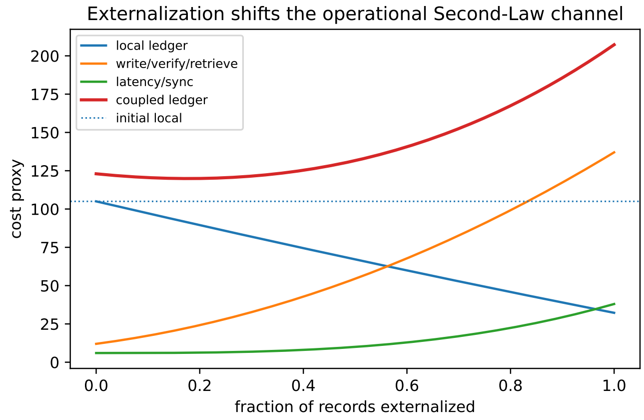

FIG. 4. Externalization shifts the operational Second-Law channel. Local cost can fall while write, verification, retrieval, latency, and synchronization make the coupled ledger rise.

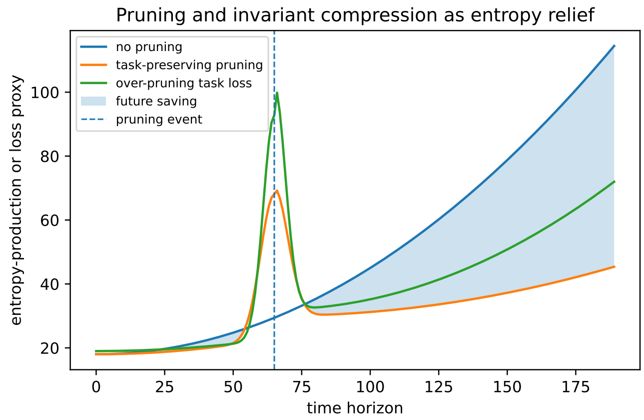

FIG. 5. Pruning and invariant compression as entropy relief. Task-preserving pruning pays a one-time cost to reduce future ledger burden; over-pruning incurs task loss.

with write, verification, retrieval, latency, synchronization, maintenance, and environmental noise. Local improvement without coupled-system improvement counts as relocation, not elimination.

Protocol 4 (Invariant compression test)**.** *Replace detailed records with a quotient or invariant representation. Measure whether task loss remains below tolerance while the erase/refresh/verification ledger decreases. A successful quotient is an entropy-relief channel; an unsuccessful quotient is over-compression.*

Protocol 5 (Finite-memory hysteresis)**.** *Raise task demand until memory reuse, pruning, erasure, or damage occurs, then reduce demand. Test whether energy, latency, and error curves return to their original path or retain a gap from record loss, damage, or lost invariant support.*

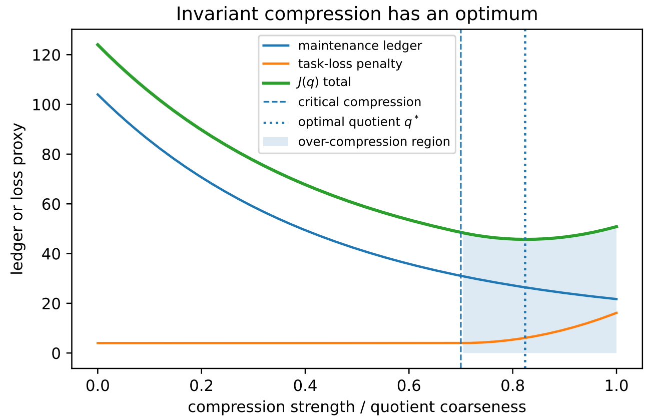

FIG. 6. Invariant compression has an optimum. Under-compression carries avoidable record burden; moderate invariant compression lowers maintenance cost; over-compression destroys task-relevant distinctions and raises task loss.

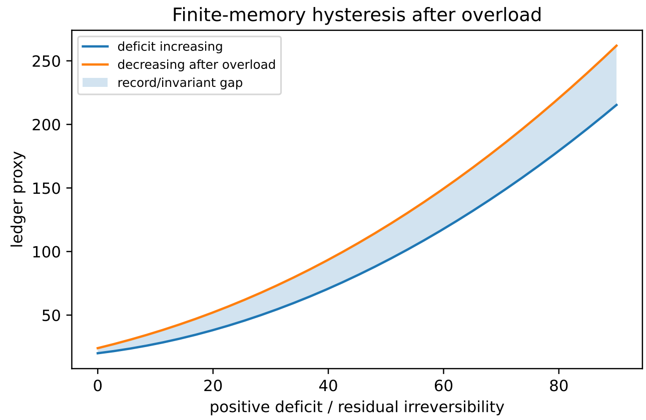

FIG. 7. Finite-memory hysteresis after overload. The gap represents record loss, accumulated damage, or destroyed invariant support, not a universal hysteresis law.

RELATION TO EXISTING WORK

Landauer's principle and Bennett's reversible computation establish the distinction between logical irreversibility, reversible embeddings, and physical implementation costs [11–13]. Modern information thermodynamics extends this framework to measurement, feedback, correlations, and stochastic trajectories [14–16]. Stochastic thermodynamics of computation emphasizes finite-time and architectural constraints [18, 19]. Recent work on realistic electronic memory erasure and quantum many-body Landauer accounting further emphasizes that record-turnover cost depends on concrete physical architectures and system-environment partitions [30, 31]. Thermodynamic uncertainty relations link precision of nonequilibrium currents to dissipation [20, 21]. Compared with the free-energy principle and active in-

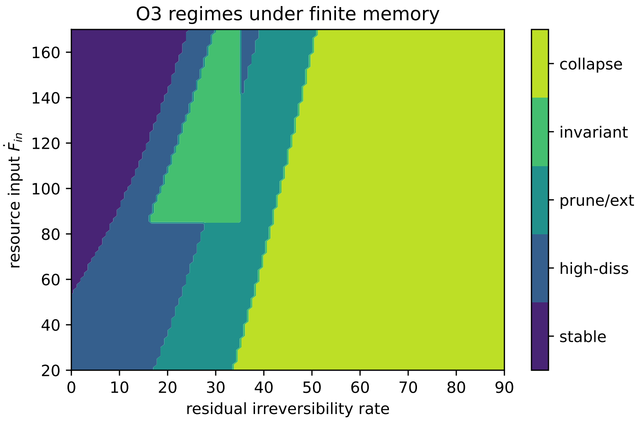

FIG. 8. O3 regimes under finite memory. Region boundaries are synthetic normal-form thresholds, not fitted phase boundaries. Stable, high-dissipation, pruning/externalization, invariant-compression, and collapse regimes arise from the relation between residual irreversibility and resource input. These regions correspond to the exit channels in Theorem 1 and Corollary 1.

ference [27, 32], O3 focuses on physical record turnover and coupled entropy/resource ledgers rather than variational free-energy objectives. O3 does not replace these theories. It packages the finite-record channel by which a boundary-maintaining system instantiates them.

Compared with P5, O3 emphasizes memory reuse and the macroscopic arrow rather than capacity-deficit correction pressure [8]. Compared with O2, O3 adds the entropy ledger to register time [3]. Compared with P2, O3 turns bounded reversible history into an operational Second-Law channel [6].

Topological persistence as quarantined projection

A separate high-risk bridge direction may treat non-Hermitian skin effects and point-gap topology as possible global-invariant constraints on coarse-graining. O3 does not use such topological claims in its core theorem. If a domain-specific topological invariant exists, O3 treats it as invariant-supported persistence or entropy relocation: it may protect an identity predicate against local erasure, but it does not license perpetual motion or global entropy-law violation. The conservative statement is that topology, when physically realized, changes which structures are maintainable and where entropy is produced; it does not remove the coupled ledger.

LIMITATIONS AND FALSIFICATION

O3 does not claim that all irreversibility is finite-memory erasure, that all coarse-graining is heat, that externalization always helps, that pruning always helps, or that invariant compression is always available. It also does not identify thermodynamic entropy with memory use. Its object is record-turnover entropy: the entropy/resource cost incurred when finite physical registers maintain a boundary by reusing, erasing, refreshing, repairing, synchronizing, or externalizing records. It is a finite-record bridge under physical implementation assumptions.

The strong version of O3 would be weakened or rejected by any of the following:

1. bounded-memory systems sustain unbounded usable record history with no reuse, compression, externalization, task relaxation, or failure;
2. physically irreversible record reuse violates the stated Landauer-style bridge assumptions;
3. stable finite records require no refresh, repair, verification, synchronization, carrier stability, or resource cost in any implementation;
4. externalization reduces local cost while imposing no coupled-system write, verification, retrieval, maintenance, latency, or environmental cost;
5. finite-memory overload produces no measurable reset, heat, latency, error floor, pruning, externalization, recovery, or collapse signature;
6. invariant compression never reduces maintenance burden in any controlled system where task identity is preserved.

CONCLUSION

FDS-O3 gives a finite-memory operational Second-Law channel. It does not claim that the Second Law is derived from distinction alone. It claims that when sustained task-relevant record demand exceeds available finite memory, continued updating requires record reuse, side records, externalization, compression, pruning, or failure. If record reuse is irreversible within an accounting boundary, the physical implementation appears in an entropy or resource ledger.

The compact statement is:

> *Finite memory turns boundary maintenance into record turnover. Physical record turnover, when irreversible or actively maintained, instantiates an operational Second-Law channel.*

This places O3 after O1 and O2 in the Operational Trident [2, 3] and after P1, P2, N1, and P5 in the physical accounting spine [5–8]. It supplies a bridge from finite records to macroscopic irreversibility while leaving the classical Second Law, stochastic thermodynamics, and domain-specific topological questions intact.

Notation summary

Symbol	Meaning
B	maintained boundary or interface
M	memory, model, or record state
Y	observation channel
U	update rule
ℓ	boundary-maintenance loss
Φ	finite resource budget
τ	update window
C_{mem}	finite memory capacity
$B_R(t;\epsilon,\Psi)$	task-relevant record load
$\rho_M(t)$	memory-fill ratio
$G_{\mathcal{A},t}$	accessible side records in boundary $\mathcal{A}$
$L_{\mathcal{A}}(t)$	one-step residual irreversibility
$r^{\mathcal{A}}_{\mathrm{res}}(t;\tau)$	finite-window residual turnover rate
$\dot{\Sigma}^{\mathcal{A}}_{\mathrm{O3}}$	audited O3 entropy ledger
$\mathcal{E}_{\mathrm{exit}}$	record-equivalent exit relief
$D_\ell(\ell)$	functional loss or damage
q	invariant quotient map

Simulation parameters

The simulations use deterministic arrays with fixed synthetic parameters in `code/generate_results.py`. Figure 1 varies record load and finite memory capacity. Figure 2 compares reversible logging, bounded cleanup, and lossy overwrite. Figure 3 decomposes the finite-record entropy ledger. Figure 4 audits externalization. Figure 5 compares pruning policies. Figure 6 compares representation strategies. Figure 7 illustrates overload hysteresis. Figure 8 classifies regimes under residual irreversibility and resource input. No proprietary, biological, organizational, quantum-experimental, human-subject, or device data are used.

Reproducibility checklist

1. Code availability: all simulation code is included in the replication package.
2. Deterministic execution: the code uses fixed synthetic parameters and deterministic arrays.
3. Figure reproduction: run `python code/generate_results.py`; the script regenerates all figures and CSV outputs.
4. Data status: all numerical outputs are synthetic demonstrations generated from the stated model.
5. Platform independence: the code uses standard Python scientific libraries.

Code availability

The simulation code used to generate Figs. 1–8 is included in the accompanying replication package under `code/generate_results.py`. Running the script regenerates all figures (PDF and PNG) and CSV tables in a single pass.

AI assistance disclosure

AI-assisted tools were used for language polishing, structural feedback, LaTeX drafting support, and code-debugging assistance. The author reviewed and edited all content and remains responsible for all claims, references, simulations, and conclusions. No AI system is listed as an author.

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